Theory of Equations

Q.) Every equation of n^{th} degree has n roots and no more.

Proof:- Let α_1 be the root of $f_n(x) = 0$; ~~there~~ from then we have $f_n(x)$ is divisible by $(x - \alpha_1)$ without remainder.

$$\therefore f_n(x) \equiv (x - \alpha_1) f_{n-1}(x)$$

where the quotient $f_{n-1}(x)$ is a polynomial of $(n-1)^{th}$ degree in x. Again, according to the fundamental theorem, $f_{n-1}(x) = 0$ has a Droot; let the root be α_2.

Then $f_{n-1}(x)$ is divisible by $(x - \alpha_2)$

$$\therefore f_{n-1}(x) \equiv (x - \alpha_2) f_{n-2}(x)$$

and Consequently from (1), $f_n(x) = (x-\alpha_1)(x-\alpha_2) f_{n-2}(x)$.

Similarly $f_{n-2}(x) = 0$ has a root, say α_3.

We then have $f_{n-2}(x) = (x - \alpha_3) f_{n-3}(x)$ where $f_{n-3}(x)$ is a polynomial of degree $n-3$ in x.

Thus, $f_n(x) \equiv (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) f_{n-3}(x)$.

Continuing this process, we shall get

$$f_n(x) \equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \cdots (x-\alpha_n)Q \quad \text{--- (2)}$$

Now each of $f_n(x)$ and $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$ is of nth degree and hence Q must be independent of x.

Now $f_n(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$

Hence equating the coefficient of x^n from both sides of (2) we get

$$Q = a_0.$$

Thus from (2),

$$f_n(x) \equiv a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n) \quad \text{--- (3)}$$

Now, the R.H.S. of (3) vanishes when $x = \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$.

The equation $f_n(x)=0$ has, therefore n roots.

Now, to prove that $f_n(x)=0$ has got n and only n roots.

Let $\delta \neq \alpha_1, \alpha_2, \cdots, \alpha_n$

Then no factor of $f_n(x)$ can vanish as is evident from (3) and consequently $f_n(x) \neq 0$ for $x = \delta$.

Hence $f_n(x)$ cannot have more than n roots.

This proves the theorem.

$f(x) = (x-a)\, q(x) + r(x)$; where $\deg. r(x) < \deg. g(x) < 1$

$= (x-a)\, q(x) + r$, where r is a Constant.

$\therefore f(a) = (a-a)\, q(x) + r$

$\therefore r = f(a)$

Therefore we get the Known result (which is Known as the Remainder theorem) that if the polynomial be divided by $(x-a)$, the remainder is $r = f(a)$

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Notes:- (i) If $f(a) = 0$, then the remainder is zero i.e. the polynomial is Completely divisible by $x-a$.

(ii) If $f_n(x)$ is divisible by $x-\alpha$, then α shall be a root of the equation $f_n(x) = 0$. Conversely, if α be a root of the equation $f_n(x) = 0$, then $f_n(x)$ shall be divisible by $(x-\alpha$