Theory of two polynomials

H.C.F. of two polynomials:

The greatest Common divisor of two polynomials $f(x)$ and $g(x)$ is that polynomial (monic polynomial) whose leading Coefficient is unity $d(x)$ which is divisor of both $f(x)$ and $g(x)$ and every divisor of both $f(x)$ and $g(x)$ is a divisor of $d(x)$.

Expression of H.C.F. as a linear Combination of integers

The Euclidian Algorithm is not only useful in Calculating the H.C.F. of two integers, but it is also useful in expressing the H.C.F of two integers as a linear Combination of these integers. In fact, each of the remainders in equation of $r_{k-1} = q_{k+1} r_k$ Can be expressed in turn, as a linear Combination of a and b.

From the first of equations (I) we see that $r = a - qb$ and hence r is a linear Combination of a and b.

From the second equation, we have $r_1 = b - q_1 r$ and substituting for r, we get

$$r_1 = b - q_1(a - qb)$$
$$= b - q_1 a + q_1 q b$$
$$= (1 + q_1 q) b - q_1 a$$

and hence r_1 is a linear combination of a and b.

Again, from the third equation, we have $r_2 = r - q_2 r_1$ and substituting for r and r_1, we get

$$r_2 = (a - qb) - q_2\{(1 + q_1 q) b - q_1 a\}$$
$$= (a - qb) - q_2(1 + q_1 q) b + q_2 q_1 a$$
$$= (1 + q_2 q_1) a - \{q + q_2(1 + q_1 q)\} b$$

so that r_2 is a linear combination of a and b.

Proceeding in this way, we shall find that each remainder and in particular the H.C.F. r_k can be expressed as a linear combination of a and b.

This can also be shown by induction by assuming that r_{k-1} is a linear combination of the form $r_{k-1} = xa + yb$, where $x, y \in I$.

For example, we consider the various steps involved in calculating the H.C.F. of 156 and 240 as in the preceding example.

Let us write $a = 240$ and $b = 156$. The calculations are as follows:

$240 = 1 \cdot 156 + 84 \qquad \therefore 84 = a - 1 \cdot b$

$156 = 84 \cdot 1 + 72 \qquad \therefore 72 = b - 84$

$= b - (a - b)$

$= 2b - a$

$84 = 72 \cdot 1 + 12 \qquad \therefore 12 = 84 - 72 \cdot 1$

$= (a - b) - (2b - a)$

$= 2a - 3b$

Hence $12 = 2(240) - 3(156)$ expressing the H.C.F. of 240 and 156 as a linear combination of these two integers

Note:-