Theory of Equations

Relations between the roots and Coefficients :-

Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ be the roots of the polynomial equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0 \quad \text{---(1)}$$

Then we have the identity

$$a_0 x^n + a_1 x^{n-1} + a_n x^{n-2} + \dots + a_{n-1} x + a_n$$
$$\equiv a_0 (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_{n-1})(x-\alpha_n)$$
$$= a_0 \{x^n - (\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n) x^{n-1}$$
$$+ (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \dots + \alpha_{n-1}\alpha_n) x^{n-2}$$
$$- (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_3\alpha_4 + \dots + \alpha_{n-2}\alpha_{n-1}\alpha_n) x^{n-3}$$
$$+ \dots + (-1)^n \alpha_1\alpha_2\alpha_3 \dots \alpha_n\}.$$

Hence equating the Coefficients of the like powers of x from both sides, we get

$$\sum \alpha_1 \equiv \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n = \frac{a_1}{a_0}$$

$$\sum \alpha_1\alpha_2 \equiv \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \dots + \alpha_{n-1}\alpha_n = \frac{a_2}{a_0}$$

$$\sum \alpha_1\alpha_2\alpha_3 \equiv \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \dots + \alpha_{n-2}\alpha_{n-1}\alpha_n = -\frac{a_3}{a_0}$$

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$$\alpha_1\alpha_2\alpha_3 \dots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

It is to be observed that the number of terms in $\sum \alpha_1 \alpha_2$ is ${}^nC_2$, for this is equal to the number of combinations of n roots taken two at a time. Similarly the number of terms in $\sum \alpha_1\alpha_2\alpha_3$ is ${}^nC_3$ and so on.

It is very important to note that before writing down the relation between the roots and the coefficients of a given equation, it must be made complete if it is not so.

Working Rule :-

Let $f(x) = 0$ be a (complete) equation of the nth degree and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be its n roots. Then we have

$$\text{Sum of the roots} = -\frac{\text{Coefficient of } x^{n-1}}{\text{Coefficient of } x^n}$$

Sum of the products of the roots taken two by two

$$= \frac{\text{Coefficient of } x^{n-2}}{\text{Coefficient of } x^n}$$

Sum of the products of the roots taken three by three

$$= \frac{-\text{Coefficient of } x^{n-3}}{\text{Coefficient of } x^n}$$

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Product of all the roots $= (-1)^n \dfrac{\text{Constant term}}{\text{Coefficient of } x^n}$

For example, let $ax^3+3bx^2+3cx+d=0$ be a Cubic equation and let α, β, γ be its roots.

Then we have, $\alpha+\beta+\gamma = -\frac{3b}{a}$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{3c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

For another example, let $3x^4-6x^3+11x^2+27x-30=0$ be a biquadratic equation, and let $\alpha, \beta, \gamma, \delta$ be its roots.

Then we have, $\alpha+\beta+\gamma+\delta = -\frac{6}{3} = 2$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{11}{3}$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = \frac{-27}{3} = -9$$

$$\alpha\beta\gamma\delta = \frac{-30}{3} = -10$$