Theory of Equations

2018

Q.) In an equation with real coefficients imaginary roots occur in conjugate pairs.

Let $\alpha+i\beta$ be a root of the polynomial equation $f(x)=0$. Then we have to prove that $\alpha-i\beta$ is also a root of the equation.

Following division algorithm, we divide the polynomial $f(x)$ by $[x-(\alpha+i\beta)]\{x-(\alpha-i\beta)\}$ i.e by $(x-\alpha)^2+\beta^2$.

Let the quotient be Q and the remainder be $Rx+R'$.

Then we have $f(x) \equiv \{(x-\alpha)^2+\beta^2\}Q+(Rx+R')$ —— (1)

Now, since $\alpha+i\beta$ is a root of the equation, therefore
$$f(\alpha+i\beta)=0.$$

Hence putting $x=\alpha+i\beta$ in (1), we get
$$f(\alpha+i\beta) \equiv R(\alpha+i\beta)+R'$$

i.e. $R(\alpha+i\beta)+R'=0$; since $f(\alpha+i\beta)=0$

$\Rightarrow R\alpha+iR\beta+R'=0$

$\Rightarrow (R\alpha+R')+iR\beta=0$

$\Rightarrow R\alpha+R'=0$ and $R\beta=0$; by equating to zero, the real part and imaginary part seperately.

Now $R\beta = 0 \Rightarrow R = 0$; since $\beta \neq 0$. otherwise the roots will be real

and $R\alpha + R' = 0 \Rightarrow R' = 0$, since $R = 0$.

$\therefore R = 0$ and $R' = 0$

Hence from (1), $f(x) \equiv \{(x-\alpha)^2 + \beta^2\} Q$

$$\equiv (x - \alpha + i\beta)(x - \alpha - i\beta) Q$$

$$\equiv \{x - (\alpha - i\beta)\}\{x - (\alpha + i\beta)\} Q$$

This shows that $x - (\alpha - i\beta)$ is a factor of $f(x)$ and Consequently $x = \alpha - i\beta$ is a root of the equation $f(x) = 0$.

Thus the theorem is proved.

Cor:- Every equation of odd degree has at least one real root.

The proof follows from the following Consideration:

Let the polynomial equation $f(x) = 0$ be of n^{th} degree where n is an odd integer. Therefore the equation $f(x) = 0$ has n roots i.e. the number of roots, real or imaginary, of $f(x) = 0$ is odd.

But we also know that the imaginary roots enter in pairs.

Hence the number of real roots of the given equation $f(x)=0$ must be an odd integer.

Thus every equation of odd degree must have at least one real root.