07/04/2021

Theorem If $S = \{v_1, v_2, v_3, \cdots v_n\}$ is an orthogonal set of non-zero vectors in an inner product space V, then S is linearly independent of V.

"or"

An orthogonal set of non-zero vectors is linearly independent.

Proof - Since $S = \{v_1, v_2, v_3, \cdots v_n\}$ then show that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0$

where $\alpha_i \in F$ implies that

$\alpha_1 = \alpha_2 = \cdots \alpha_n = 0$ for L.I.

$$\sum_{i=1}^{n} \alpha_i v_i = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0 \quad \text{--(I)}$$

From the inner product space

$$\left\langle \sum_{i=1}^{n} \alpha_i v_i , v_i \right\rangle = \langle 0, v_i \rangle$$

$$\alpha_1 \langle v_1, v_i \rangle + \alpha_2 \langle v_2, v_i \rangle + \cdots \alpha_i \langle v_i, v_i \rangle$$
$$+ \cdots \alpha_n \langle v_n, v_i \rangle = \langle 0, v_i \rangle \quad \text{--(II)}$$

S is orthogonal than $\langle v_i, v_j \rangle = 0$ for $i \neq j$

then Eqn (II) becomes

$$\alpha_i \langle v_i, v_i \rangle = \langle 0, v_i \rangle = 0$$

But each vector in S is non-zero i.e $v_i \neq 0$ then $\langle v_i, v_i \rangle \neq 0$

So $\alpha_i = 0$ then the set S is L.I.

Theorem An orthonormal set of vectors is linearly independent (L.I.) of vector space V.

Proof Let $S = \{v_1, v_2, \ldots v_n\}$ be an orthonormal/orthogonal set of vectors in an inner product space.

To prove that S is L.I. as orthonormal conditions

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & i \neq j \end{cases}$$

Suppose S is Linear independent then

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0$$

$$\alpha_i \in F \quad \text{and } \alpha_i = 0$$

$$\sum_{i=1}^{n} \alpha_i v_i = 0 \quad \text{——} (1)$$

Form of inner product space from Eq (1)

$$\langle \sum_{i=1}^{n} \alpha_i v_1, v_j \rangle = \langle 0, v_j \rangle$$

$$\alpha_1 \langle v_1, v_j \rangle + \alpha_2 \langle v_2, v_j \rangle + \cdots \text{——} \alpha_i \langle v_i, v_j \rangle$$

$$+ \cdots \alpha_n \langle v_n, v_j \rangle = 0 \quad \text{——} (11)$$

By orthonormal conditions

$\langle v_i, v_j \rangle = 0$ if $i \neq j$ & $\langle v_j, v_j \rangle = 1$ if $i=j$

then Eqn (2) be comes as

$\alpha_j \langle v_j, v_j \rangle = 0$

$\alpha_i \cdot 1 = 0 \qquad \because \langle v_j, v_j \rangle = 1$

$\alpha_i = 0$

$\therefore \alpha_1 = 0, \alpha_2 = 0 \ldots \alpha_n = 0$

Hence orthonormal set of vectors is linearlyindependent.