Abstract Algebra

SOME THEOREMS ON SEPARABILITY

Theorem :- Every field of characteristic zero is perfect.

Proof :- Let K be any finite extension of a field F of characteristic zero, then F will be perfect if all the finite extensions of F are separable.

For this we shall prove that every element of K is separable, i.e. we shall show that the minimal polynomial for each element of K over F is separable.

Let α be an arbitrary element of K and let

$$f(x) = a_0 + a_1x + \dots + a_nx^n, \text{ with } a_n \neq 0$$

and a_i's $\in F$ be a minimal polynomial for α over F. Then

$$f'(x) = 0 \Rightarrow a_1x + 2a_2x + \dots + na_nx^{n-1} = 0$$

We suppose $f(x)$ has multiple root in K. then by theorem, we have $f'(x) = 0$

$$\Rightarrow a_1 + 2a_2x + \dots + na_nx^{n-1} = 0$$

$$\Rightarrow a_1 = 0,\ 2a_2 = 0, \dots\ na_n = 0 \quad \text{—— (1)}$$

But F is of characteristic zero, then (1) is possible only if $a_1 = 0 = a_2 = \dots + a_n$, this gives $f(x) = a_0$, which is a constant polynomial having a multiple root. But this gives a contradiction because no constant polynomial has a root. Hence $f(x)$ has no multiple root, so is seperable.

Therefore α is seperable over F.

Since α is an arbitrary element of K, thus every element of K is seperable over F. according K is a seperable extension of F. Also K is taken as arbitrary finite extension of F. Hence all the finite extensions of F are seperable. Consequently F is perfect.

__Theorem__ :- An irreducible polynomial $f(x)$ over a field F of characteristic $p>0$ is inseperable if and only if $f(x)=g(x^p)$, i.e. $f(x)$ is a polynomial in x^p.

__Proof__ :- let $f(x) = a_0 + a_1x + \cdots + a_nx^n$, with $a_n \neq 0$ be an irreducible polynomial over a field F of characteristic $p>0$. Then

$$f'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$$

Suppose that $f(x)$ is inseperable, then $f(x)$ has at least one multiple root in its splitting field over F.

Now by the theorem [of multiple root that for any two polynomials $f(x)$ and $g(x)$ in $F[x]$,

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

And if $f(x)$ an irreducible polynomial over a field F. Then $f(x)$ has a multiple root in some field extension if and only if $f'(x)=0$.]. we have

$$f'(x) = 0 \Rightarrow a_1 + 2a_2x + \cdots + a_nx^{n-1} = 0$$

$$\Rightarrow a_1 = 0,\ 2a_2 = 0, \cdots na_n = 0$$

$$\Rightarrow ra_r = 0,\ 0 \leq r \leq n.$$

Since F is a field of characteristic $p>0$, then

$$r a_r = 0 \Longrightarrow \text{either } a_r = 0 \text{ or, } p \mid r$$

$$\Rightarrow \text{either } a_r = 0 \text{ or if } a_r \neq 0.$$

then $r = kp$ for some positive k.

This $f(x) = b_0 + b_1 x^p + b_2 x^{2p} + \cdots + b_m x^{mp}$, where $b_j = a_{jp}$.

Consequently, if $f(x)$ is a polynomial in x^p over F of characteristic $p>0$, then

$$f'(x) = 0.$$

Let if possible $f(x)$ has no multiplex root, then $f'(x) = 0$ for any x, which gives a contradiction.