17/05/2021 TDC-III

Theorem If set $S=\{v_1, v_2, --- v_n\}$ is a basis of $V(F)$, then every elements (vectors) of V can be uniquely expressible as linear combination of v_1, v_2, v_n.

Proof Since $V=L(S)$, each $v \in V$ is expressible as $v=\alpha_1 v_1+\alpha_2 v_2+ \cdots\cdots \alpha_n v_n$ — ①

$\alpha_i \in F$ & $v_i \in V$

This Equation (1) is unique expressible

$$v=\beta_1 v_1+\beta_2 v_2+\cdots\cdots \beta_n v_n \text{ — ②}$$

$\beta_i \in F$ & $v_i \in V$

From Eqn (1) and (11)

$$\alpha_1 v_1+\alpha_2 v_2+\cdots - \alpha_n v_n=\beta_1 v_1+\beta_2 v_2+\cdots - \beta_n v_n$$

$$\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n-\beta_1 v_1-\beta_2 v_2-\cdots\cdots-\beta_n v_n=0$$

$$(\alpha_1-\beta_1)v_1+(\alpha_2-\beta_2)v_2+\cdots\cdots(\alpha_n-\beta_n)v_n=0$$

The set $S=\{v_1, v_2, \cdots\cdots v_n\}$ is basis of V.

So $v_1, v_2, -v_n$ are Linear independent.

$\alpha_1-\beta_1=0 \Rightarrow \alpha_1=\beta_1$

$\alpha_2-\beta_2=0 \Rightarrow \alpha_2=\beta_2$

$\cdots \qquad \cdots$

$\cdots \qquad \cdots$

$\alpha_n-\beta_n=0 \Rightarrow \alpha_n=\beta_n$

Hence Eqn (1) is a unique expression of v as linear combination of $S=\{v_1, v_2 \cdots v_n\}$

Theorem – Every subset of a linear independent set is linearly independent.

Proof Let $T = \{v_1, v_2, \dots v_m, v_{m+1}, \dots v_n\}$ be linearly independent vectors.

But $S \subseteq T$ then $S = \{v_1, v_2 - - v_m\}$

$1 \le m \le n$

Let the set of vectors S is L.I then

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_m v_m = 0$$

$\alpha_i \in F$ & $v_i \in S$

$i = 1$ to m

Now

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_m v_m + 0 \cdot v_{m+1} + 0 \cdot v_{m+2} + \dots$$

$$\dots \quad 0 \cdot v_n = 0$$

i.e $\alpha_{m+1} = 0$, $\alpha_{m+2} = 0 \dots \alpha_n = 0$

$\therefore$ The set $T = \{v_1, v_2, \dots v_m, v_{m+1}, \dots v_n\}$ be L.I

then $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_m v_m = 0$

$$\Rightarrow \alpha_1 = 0, \ \alpha_2 = 0 \dots \alpha_m = 0$$

Hence the set $S \subseteq T$ is L.I.

Theorem - A subset of a linearly dependent set is linearly dependent.

Proof: Let $S = \{v_1, v_2, v_3 \dots v_r\}$ be Linearly dependent set of vectors and $T = \{v_1, v_2, v_3 \dots v_r, v_{r+1}, \dots v_n\}$ be a superset of S.

Since S is L.D. there exist scalers $\alpha_1, \alpha_2, \dots \alpha_r$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_r v_r = 0$$

or $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_r v_r + 0 \cdot v_{r+1} + 0 \cdot v_{r+2} \dots 0 \cdot v_n = 0$

where $\alpha_1, \alpha_2, \dots \alpha_r$ are not all zero

Hence $T = \{v_1, v_2, \dots v_r, \dots v_n\}$ is L.D

$\Rightarrow$ S is L.D.

Problem - Let $V(F)$ be a vector space and a subset $S_1 = \{v_1, v_2, \dots v_n\}$ of $V(F)$ be a L.I. set. If $v \in V(F)$ and $v \notin L(S)$, then show that $S_2 = \{v, v_1, v_2 \dots v_n\}$ is L.I set.

Sol. let $\alpha v + \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0$

α and $\alpha_i \in F$

If $\alpha \neq 0 \in F$, then $\alpha^{-1} \in F$ and $\alpha \alpha^{-1} = \alpha^{-1}\alpha = 1$

$$\alpha^{-1}(\alpha v + \alpha_1 v_1 + \dots \alpha_n v_n) = \alpha^{-1} 0$$

$$\alpha^{-1}\alpha v + \alpha^{-1}\alpha_1 v_1 + \alpha^{-1}\alpha_2 v_2 + \dots \alpha^{-1}\alpha_n v_n = 0$$

$$v = (-\alpha^{-1}\alpha_1) v_1 + (-\alpha^{-1}\alpha_2) v_2 + \dots (-\alpha^{-1}\alpha_n) v_n$$

This implies that $v \in L(S)$, which is contradiction

$\therefore \alpha = 0$ and so $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0 \dots \alpha_n = 0$ So S_1 and S_2 are L.I.