Theorem:- Every n-dimensional vector space $U(F)$ is isomorphic to $V_n(F)$

or

Show that every n-dimensional vector space $U(F)$ is isomorphic to $V(F^n)$.

$\because T: U(U) \to V_n(F)$

Proof Let $\{u_1, u_2, \ldots u_n\}$ be any basis of $U(F)$

Then every vector $u \in U$ can be uniquely expressed as-

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n \qquad \alpha_i \in F$$
$$i = 1, 2 \ldots n$$

$$T(u) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots \alpha_n T(u_n)$$

The ordered n-tuple/dimension

$$\{\alpha_1, \alpha_2, \ldots \alpha_n) \in V_n(F)$$

Let $T: U(F) \to V_n(F)$ be defined by

$$T(u) = (\alpha_1, \alpha_2, \ldots \alpha_n)$$

$\because u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n$

Since in the expression of u as a linear combination of $u_1, u_2 \ldots u_n$ the scalars $\alpha_1, \alpha_2, \ldots \alpha_n$ are unique, therefore $T(u)$ is a unique element of $V_n(F)$ and thus the mapping T is well-defined

(1) T is one-one:- let $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots \alpha_n u_n$

and $u' = \beta_1 u_1 + \beta_2 u_2 + \ldots \beta_n u_n$ be any two elements of U.

We have $T(u) = T(u')$

$\Rightarrow T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n) = T(\beta_1 u_1 + \beta_2 u_2 + \cdots \beta_n u_n)$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \cdots \beta_n v_n$

$(\alpha_1, \alpha_2, \cdots \alpha_n) = (\beta_1, \beta_2, \cdots \beta_n)$

$\Rightarrow u = u'$

$\therefore$ T is one-one.

(ii) T is onto $V_n(F)$: let $(\alpha_1, \alpha_2, \cdots \alpha_n)$ be any element of F or $V_n(F)$ such that

$T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n) = (\alpha_1, \alpha_2, \cdots \alpha_n)$

T is onto $V_n(F)$

(iii) <u>T is Linear Transformation</u>:

If $\alpha, \beta \in F$ and $u, u' \in U(F)$

we have $T(\alpha u + \beta u')$

$= T(\alpha(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n) + \beta(\beta_1 u_1 + \beta_2 u_2 \cdots \beta_n u_1))$

$= T[(\alpha\alpha_1 + \beta\beta_1)u_1 + (\alpha\alpha_2 + \beta\beta_2)u_2 + \cdots (\alpha\alpha_n + \beta\beta_n)u_n]$

$= \{(\alpha\alpha_1 + \beta\beta_1), (\alpha\alpha_2 + \beta\beta_2), \cdots (\alpha\alpha_n + \beta\beta_n)\}$

$= \alpha(\alpha_1 + \alpha_2 + \cdots \alpha_n) + \beta(\beta_1 + \beta_2 + \cdots \beta_n)$

$= \alpha T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n) + \beta T(\beta_1 u_1 + \beta_2 u_2 + \cdots \beta_n u_n)$

$= \alpha T(u) + \beta T(u')$

$\therefore$ T is Linear Transformation

$\therefore$ T is an isomorphism of $U(F)$ onto $V_n(F)$

Hence $U(F) \cong V_n(F)$

i.e. $U(F)$ is isomorphic to $\underline{\underline{V_n(F)}}$