Theorem Let V be an inner product space and $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent subset of V. Defines $S' = \{w_1, w_2, \dots, w_n\}$ where $w_1 = v_1$ and

$$w_k = v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, w_j \rangle}{\|w_j\|} w_j \quad \text{for } 2 \le k \le n \quad \text{---(1)}$$

Then S' is an orthogonal set of nonzero vectors such that $\text{span}(S') = \text{span}(S)$.

Proof The proof is by mathematical induction on n, the number of vectors in S. For $k = 1, 2, \dots, n$ let $S_k = \{v_1, v_2, \dots, v_k\}$ be subset of basis for V.

If $n = 1$ then $S_1' = S_1$ i.e. $w_1 = v_1 \ne 0$

Assume then that the set $S_{k-1}' = \{w_1, w_2, \dots, w_{k-1}\}$ with the desired properties has been constructed by the repeated use of (1).

We show that the set $S_k' = \{w_1, w_2, \dots, w_{k-1}, w_k\}$ also has the desired properties, where w_k is obtained from S_{k-1}' by (1).

If $w_k = 0$, then (1) implies that $w_k \in \text{span}(S_{k-1}') = \text{span}(S_{k-1})$, which contradicts the assumption that S_k is L.I.

For $1 \le i \le k-1$, it follows from (1) that

$$\langle w_k, w_i \rangle = \langle v_k, w_i \rangle - \sum_{j=1}^{k-1} \frac{\langle v_k, w_j \rangle}{\|w_j\|^2} \langle w_j, w_i \rangle$$

$$= \langle w_k, w_i \rangle - \frac{\langle v_k, w_i \rangle}{\|w_i\|^2} \|w_i\|^2 = 0$$

Since $\langle w_j, w_i \rangle = 0$ if $i \ne j$ by the induction assumption that S_{k-1}' is orthogonal.

Hence S_k' set of vectors is an orthogonal set basis of non zero vectors

We have that $\text{span}(S_k') \subseteq \text{span}(S_k)$

S_k' is L.I so $\dim(\text{span}(S_k')) = \dim(\text{span}(S_k)) = k$

Therefore $\text{span}(S_k') = \text{span}(S_k)$

The construction of $\{w_1, w_2, \dots, w_n\}$ by use of this theorem is called Gram–Schmidt process

Exp. In R^4 the set $\{v_1(1,0,1,0), v_2(1,1,1,1)$ and $v_3(0,1,2,1)\}$ be basis of V. Then $\{v_1, v_2, v_3\}$ is L.I. We use the Gram-Schmidt process to compute (construct) the orthogonal vectors w_1, w_2, w_3 and then we normalize these constructed vectors to obtain an orthonormal set.

Sol. Take $w_1 = v_1 = (1,0,1,0)$

Then construction of orthogonal vector by using G.S. process,
$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} \cdot w_1$$

$$\langle v_2, w_1\rangle = (1,1,1,1)(1,0,1,0) = 1+0+1+0 = 2$$

$$\|w_1\|^2 = \langle w_1, w_1\rangle = (1,0,1,0)(1,0,1,0) = 1+0+1+0 = 2$$

$$w_2 = (1,1,1,1) - \frac{2}{2}(1,0,1,0)$$

$$= (1,1,1,1) - (1,0,1,0) = (0,1,0,1)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} \cdot w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} \cdot w_2$$

$$= (0,1,2,1) - \frac{(0+0+2+0)}{2}(1,0,1,0) - \frac{(0+1+0+1)}{(0+1+0+1)}(0,1,0,1)$$

$$= (0,1,2,1) - (1,0,1,0) - (0,1,0,1)$$

$w_3 = (-1,0,1,0)$ These vectors can be normalized to obtain basis (u_1, u_2, u_3)

$$u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1,0,1,0) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{2}}(0,1,0,1) = \left(0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$u_3 = w_3/\|w_3\| = \frac{1}{\sqrt{2}}(-1,0,1,0) = \left(-1/\sqrt{2}, 0, \frac{1}{\sqrt{2}}, 0\right)$$