Theorem – Every basis for a finite-dimensional vector space must contain the same number of vectors.

or **Theorem** – Any two bases of a finite dimensional vector space have the same number of vectors (elements).

Proof Vector space V is a finite dimensional vector space, V has a basis.

Let $S = \{v_1, v_2, \dots v_n\}$ and $T = \{w_1, w_2, w_3, \dots w_m\}$ be any two set of bases of V.

$\Rightarrow$ We prove that $n = m$

Suppose $n > m$. Since $v_i \in V$ and T is basis of V ~~$S = \{v_1, v_2 \dots v_n\}$ be any subset of V.~~

there exist $\alpha_{ji} \in F$ ~~and~~

Such that

$$v_i = \alpha_{1i} w_1 + \alpha_{2i} w_2 + \dots \alpha_{mi} w_m \quad \text{①} \quad \begin{matrix} i = 1, 2, \dots n \\ j = 1, 2, \dots m \end{matrix}$$

$$\left.\begin{aligned}
v_1 &= \alpha_{11} w_1 + \alpha_{21} w_2 + \dots \alpha_{m1} w_m \\
v_2 &= \alpha_{12} w_1 + \alpha_{22} w_2 + \dots \alpha_{m2} w_m \\
&\vdots \\
v_n &= \alpha_{1n} w_1 + \alpha_{2n} w_2 + \dots \alpha_{mn} w_m
\end{aligned}\right\}$$

S is basis of V then S is L. Independent.

* Consider

$$\beta_1 v_1 + \beta_2 v_2 + \dots \beta_n v_n = 0 \qquad \beta_i \in F \quad \text{(II)}$$

Substituting each v_i from (1) to in Eqn (II)

$$\beta_1(\alpha_{11} w_1 + \alpha_{21} w_2 + \dots \alpha_{m1} w_m) + \beta_2(\alpha_{12} w_1 + \alpha_{22} w_2 + \dots \alpha_{m2} w_m) + \dots \beta_n(\alpha_{1n} w_1 + \alpha_{2n} w_2 + \dots \alpha_{mn} w_m) = 0$$

$$(\beta_1 \alpha_{11} + \beta_2 \alpha_{12} + \dots \beta_n \alpha_{1n}) w_1 + (\beta_1 \alpha_{21} + \beta_2 \alpha_{22} + \dots \beta_n \alpha_{2n}) w_2 + \dots (\beta_1 \alpha_{m1} + \beta_2 \alpha_{m2} + \dots \beta_n \alpha_{mn}) w_m = 0 \quad \text{③}$$

Since $w_1, w_2, w_3 \dots w_m$ are basis of T of V, they are Linearly independent then $w_i \neq 0 \quad i = 1, 2, \dots m$

Then Eqn ③ gives –

$$\alpha_{11}\beta_1 + \alpha_{12}\beta_2 + \alpha_{13}\beta_3 + \cdots \alpha_{1n}\beta_n = 0$$
$$\alpha_{21}\beta_1 + \alpha_{22}\beta_2 + \alpha_{23}\beta_3 + \cdots \alpha_{2n}\beta_n = 0$$
$$\cdots \qquad \cdots$$
$$\alpha_{m1}\beta_1 + \alpha_{m2}\beta_2 + \cdots \alpha_{mn}\beta_n = 0$$

This homogeneous system has m linear Eq^n and n variables therefore linear Eq^n $(m) <$ variables (n). As this, the system has a non-zero solution i.e $\beta_1, \beta_2, \cdots \beta_n$ in F not all zero. Therefore $v_1, v_2, \cdots v_n$ are linearly dependent which is a contradiction. Thus our assumption $n > m$ is wrong and so $n \leq m$. — (A) for S is L. Ind.

⇒ Suppose $m > n$, then similarly, $w_j \in V$ and using that S is basis of V there exist $\alpha_{ij} \in F$ such that

$$W_j = \alpha_{1j}v_1 + \alpha_{2j}v_1 + \cdots \alpha_{nj}v_n \quad \forall \text{ for } j = 1, 2, \cdots m, \; i = 1, 2 \cdots n$$

Consider, T is basis of V then relation

$$\beta_1 w_1 + \beta_2 w_2 + \cdots \beta_m w_m = 0 \qquad \beta_i \in F \text{ — (3)}$$

substituting w_i in Eqn

Then we find n equations and m variables and $n < m$ therefore system has a non-zero solution. Consequent $w_1, w_2, \cdots w_m$ are L.D which is contradiction. Thus we can ~~prove~~ $m > n$ is wrong and so $m \leq n$ — (B) for T is L. Ind.

from A & B Hence $n = m$ and so the two bases of V have the same number of vectors.