Imp

Theorem - An inner product space is a normed vector space but not conver-sely. (i.e. Every normed vector space is not an inner product space).

Proof - Let $V(F)$ be an inner product space.

Let $u, v \in V$ and $\alpha \in F$

We know $\|u\| = \sqrt{\langle u,u \rangle}$ → Norm of u

But $\langle u,u \rangle$ is defined on V

$\therefore$ $\|u\|$ is defined on V.

(i) $\langle u,u \rangle \geq 0$ and $\langle u,u \rangle = 0$ iff $u=0$

Consequently $\|u\| \geq 0$ and $\|u\| = 0$ iff $u=0$

(ii) $\|\alpha u\| = |\alpha| \, \|u\|$

For $\|\alpha u\|^2 = \langle \alpha u, \alpha u \rangle = \alpha \langle u, \alpha u \rangle$

$= \alpha \bar{\alpha} \langle u,u \rangle$

$\|\alpha u\|^2 = |\alpha|^2 \|u\|^2$

Taking square root

$\|\alpha u\| = |\alpha| \, \|u\|$

(iii) $\|u+v\| \leq \|u\| + \|v\|$

$\therefore$ $\|u+v\|^2 = \langle u+v, u+v \rangle$

$= \langle u, u+v \rangle + \langle v, u+v \rangle$

$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$

$$\|u+v\|^2 = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + \underline{\langle u,v\rangle} + \underline{\langle v,u\rangle} + \|v\|^2$$

$$= \|u\|^2 + \|v\|^2 + 2 \text{ real part of } \langle u,v\rangle$$

$$\leq \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle|$$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$$ by Schwarz In-equality.

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \qquad \because |\langle u,v\rangle| = \|u\|\|v\|$$

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$$\therefore \|u+v\| \leq \|u\| + \|v\|$$

Thus the norm defined on an inner product space satisfies all the conditions of a normed vector space.

Hence an inner product space is a normed vector space

* But the converse is not true i.e every normed vector space is not an inner product space. For Example.

Consider a normed vector space $R^2(R)$ with a norm defined by

$$\|u\| = \max.(|u_1|, |u_2|)$$

where $u = (u_1, u_2)$

It is impossible to define an inner product by $\langle \; \rangle$ on $R^2(R)$ such that $\langle u,u\rangle = \underline{\|u\|^2}$