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Sylvester's Law

Theorem: Let U and V be vector spaces over a field F and let T be a linear transformation from U into V i.e $T: U \to V$. If U is finite-dimensional then

$$\dim(U) = \operatorname{rank}(T) + \operatorname{nullity}(T)$$

or $$\dim U = \rho(T) + \eta(T)$$

Proof Let $\dim U = n$ and $\dim N(T) = m$ i.e $\eta(T) = m$ or $\dim \ker(T) = m$

Let $S_1 = \{u_1, u_2, \dots u_m\}$ be basis of $\ker(T) \subseteq U$

Since $u_i \in \ker(T)$ for $i = 1, 2, \dots m$

$\therefore T(u_i) = 0$ —— (i)

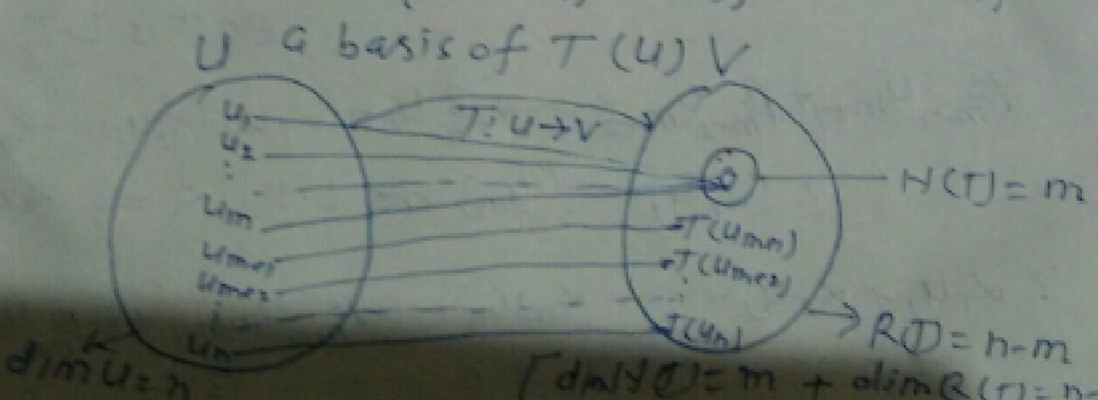
Therefore, $S_1 = \{u_1, u_2, \dots u_m\}$ are linear independent (L.I.) vectors in U. They can be extendended for form a basis of U i.e there exist vectors.

$u_{m+1}, u_{m+2}, \dots u_n$ in U.

$\therefore S_2 = \{u_1, u_2, \dots u_m, u_{m+1}, \dots u_n\}$ is a basis of U.

So, we show that

$S_3 = \{T(u_{m+1}), T(u_{m+2}), \dots T(u_n)\}$ is a basis of $T(U)$



$[\dim N(T) = m + \dim R(T) = n-m]$

Let $v \in T(U)$ be arbitrary. Then there exist some $u \in U$ such that $v = T(u)$
since S_2 is basis of U vector space then

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m + \alpha_{m+1} u_{m+1} \cdots \alpha_n u_n$$

$\alpha_i \in F$ (scalars) for $i = 1, \cdots n$

$$\because T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m + \alpha_{m+1} u_{m+1} \cdots \alpha_n u_n)$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots \alpha_m T(u_m) + \alpha_{m+1} T(u_{m+1}) + \cdots \alpha_n T(u_n)$$ as T is L.T.

$$\because T(u) = 0 + \alpha_{m+1} T(u_{m+1}) + \alpha_{m+2} T(u_{m+2}) + \cdots \alpha_n T(u_n)$$

This prove that

$T(U) = L(S_3)$

Using ①
$T(u_i) = 0$
for $i = 1, 2 \cdots m$

Now we show that S_3 is L. I. subset of U.

Let

$$\beta_{m+1} T(u_{m+1}) + \beta_{m+2} T(u_{m+2}) + \cdots + \beta_n T(u_n) = 0$$

$$T(\beta_{m+1} u_{m+1} + \beta_{m+2} u_{m+2} + \cdots + \beta_n u_n) = 0$$

$$\beta_{m+1} u_{m+1} + \beta_{m+2} u_{m+2} + \cdots + \beta_n u_n = 0 \quad \text{--- ②}$$

As T is L.T.

$$\beta_{m+1} u_{m+1} + \beta_{m+2} u_{m+2} + \cdots + \beta_n u_n = \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m$$

$\because \{u_1, u_2, \cdots u_m\}$ are L.I $\alpha_i = 0$ $\alpha_i \in F$

$$\because \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m + (-\beta_{m+1} u_{m+1}) + (-\beta_{m+2} u_{m+2}) + \cdots (-\beta_n u_n) = 0 \quad \text{--- ③}$$

$\because \alpha_1 = \alpha_2 = \alpha_3 \cdots \alpha_m = 0,\ \beta_{m+1} = \beta_{m+2} = 0,$

$\beta_{m+3} = \beta_{m+4} \text{-----} \beta_n = 0$

Since S_2 is a Linear Independent.
from ② and (3), it follow that S_3 is L.I. subset of $T(U)$.
Thus S_3 is basis of $T(U)$ as $(n-m)$

$$\dim T(U) = \dim(U) - \dim \ker(T)$$

$$\dim(U) = \dim T(U) + \dim N(T)$$

$$\dim(U) = \text{rank}(T) + \text{nullity}(T)$$

$$\dim U = \rho(T) + \eta(T)$$

Hence proved