Taylor's Series & Maclaurin's series

Q. State & prove Taylor's series

Statement:- Under certain circumstances if the function $f(x+h)$ can be expanded in a series of powers of h, then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \frac{h^3}{\underline{|3}} f'''(x) + \cdots + \frac{h^n}{\underline{|n}} f^n(x) + \cdots \infty$$

Proof:- It is given that $f(x+h)$ can be expanded in a series of integral powers of h

Therefore let $f(x+h) = A_0 + A_1 \frac{h}{\underline{|1}} + A_2 \frac{h^2}{\underline{|2}} + A_3 \frac{h^3}{\underline{|3}} + A_4 \frac{h^4}{\underline{|4}} + \cdots \infty$ — (1)

where $A_0, A_1, A_2, A_3, A_4, \cdots$ are functions of x only and are to be determined

Now $\frac{d}{dh} f(x+h) = \frac{d f(x+h)}{d(x+h)} \frac{d(x+h)}{dh} = f'(x+h)$, since x & h are independent quantities & so x may be considered constant in differentiating w.r.t. h

Similarly $\frac{d^2}{dh^2} f(x+h) = f''(x+h)$, $\frac{d^3}{dh^3} f(x+h) = f'''(x+h)$ & so on.

Differentiating both sides of (1) w.r.t. h successively & using the above differentiations, we get

$f'(x+h) = 0 + A_1 + A_2 h + A_3 \frac{h^2}{\underline{|2}} + A_4 \frac{h^3}{\underline{|3}} + \cdots$ — (2)

$f''(x+h) = A_2 + A_3 h + A_4 \frac{h^2}{\underline{|2}} + \cdots$ — (3)

$f'''(x+h) = A_3 + A_4 h + \cdots$ — (4)

$\cdots\cdots\cdots$

Putting $h=0$ in (1), (2), (3), (4), $\cdots$, we get

$A_0 = f(x)$, $A_1 = f'(x)$, $A_2 = f''(x)$, $A_3 = f'''(x)$, $\cdots$

Substituting these values in (1), we get

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}} f''(x) + \frac{h^3}{\underline{|3}} f'''(x) + \cdots + \frac{h^n}{\underline{|n}} f^n(x) + \cdots \infty$$

Q. State & prove Maclaurin's Series

Statement :- Under certain circumstances if the Series $f(x)$ can be expanded in powers of x, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots \infty$$

Proof:-

It is given that $f(x)$ can be expanded in a Series of integral powers of x, therefore

let $f(x) = A_0 + A_1 x + A_2 \frac{x^2}{\lfloor 2} + A_3 \frac{x^3}{\lfloor 3} + A_4 \frac{x^4}{\lfloor 4} + \cdots \infty$ — (1)

Where $A_0, A_1, A_2, A_3, A_4, \ldots$ are constants not containing x and are to be determined.

Differentiating (1) w.r.t. x, we get

$f'(x) = A_1 + A_2 x + A_3 \frac{x^2}{\lfloor 2} + A_4 \frac{x^3}{\lfloor 3} + \cdots \infty$ — (2)

Also, $f''(x) = A_2 + A_3 x + A_4 \frac{x^2}{\lfloor 2} + \cdots \infty$ — (3)

$f'''(x) = A_3 + A_4 x + \cdots \infty$ — (4)

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Putting $x = 0$ in (1), (2), (3), (4), ..., we get

$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = A_2,\ f'''(0) = A_3, \ldots$

Substituting these values in (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots \infty$$