Existence of Subgroups

Sylow's First theorem

Let G be a finite group and let p be a prime. If p^k divides (G), then G has at least one ~~order~~ subgroup of order p^k.

Proof:- we proceed by induction on $|G|$. If $|G| = 1$. This theorem is trivially true. Now assume that the statement is true for all groups of order less than $|G|$. If G has a proper subgroup H such that p^k divides $|H|$, then, by our inductive assumption, H has a subgroup of order p and we are done. Thus, we may henceforth assume that p^k does not divide the order of any proper subgroup of G. Next, Consider the class equation for G in the form

$$|G| = |Z(G)| + \sum |G : C(a)|,$$

where we sum over a representative of each Conjugacy class $cl(a)$, where $a \notin Z(G)$. Since p^k divides $|G| = |G : C(a)||C(a)|$ and p^k does not divide $|C(a)|$, we know that p must divide $|G : C(a)|$ for all $a \notin Z(G)$. It then follows from the class equation that p divides $|Z(G)|$. The Fundamental theorem of Finite Abelian groups

[Every finite Abelian group is a direct product of cyclic groups of prime power order. Moreover the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group.] then guarantees that $Z(G)$ contains an element of order p, say x. Since x is in the center of G, $\langle x \rangle$ is a normal subgroup of G, and we may form the factor group $G/\langle x \rangle$. Now observe that p^{k-1} divides $|G\langle x \rangle|$. Thus, by the induction hypothesis, $G/\langle x \rangle$ has a subgroup of order p^{k-1} and ~~by~~ this subgroup has the form $H/\langle x \rangle$, where H is a subgroup of G. Finally, we note that $|H/\langle x \rangle| = p^{k-1}$ and $|\langle x \rangle| = p$ imply that $|H| = p^k$. Thus we have produced a subgroup of order p^k, which contradicts our assumption that no such subgroup exists. Therefore, we must have originally had a subgroup of order p^k, and we can apply the induction hypothesis to that subgroup.

Sylow's First theorem

say we have a group G of order $2^3 \cdot 3^2 \cdot 5^4 \cdot 7$. Then Sylow's First theorem says that G must have at least one subgroup of each of the following orders: 2, 4, 8, 3. 9, 5, 25, 125, 625 and 7. On the other hand Sylow's First Theorem tells us nothing about the possible existence of subgroups of order 6, 10, 15, 30 or any other divisor of $|G|$ that has two or more distinct prime factors. Because certain subgroups guaranteed by sylow's First Theorem "Sylow p- subgroup".

[Sylow p- Subgroup :-

Let G be a finite group and let p be a prime. If p^k divides $|G|$ and p^{k+1} does not divide $|G|$, then any subgroup of G of order p^k is called a Sylow p-subgroup of G.]

So for our group G of order $2^3 \cdot 3^2 \cdot 5^4 \cdot 7$, we call ~~all~~ any subgroup of order 8 a Sylow 2-subgroup of G, any subgroup of order 625 a Sylow 5-subgroup of G, and so on. We notice that a Sylow p- subgroup of G is a subgroup whose order is the largest power of p consistent with Lagrange's