Subspace of a vector space - A non-empty subset W of a vector space V over the field F is called a subspace of V if W is a vector space under the operations/compositions/axioms of addition and scalar multiplication defined in V.

(i) if w_1 and w_2 are in W, then w_1+w_2 is in W i.e $w_1+w_2 \in W$

(ii) if w is in W and α is any scalar of field then αw is in W i.e $\alpha w \in W$

Theorem.

* If W is a nonempty subset of vector space V, then W is a subspace of V iff the following conditions hold.
(i) $w_1+w_2 \in W \ \forall w_1, w_2 \in W$ (ii) $\alpha w \in W \quad w \in W \ \& \ \alpha \in F$

$\Rightarrow \alpha w_1 + \beta w_2 \in W \quad w_1, w_2 \in W \ \& \ \alpha, \beta \in F$ (scalars)

Exp. Show that the set $W = \{(x_1, 0, x_3) : x_1 \text{ and } x_3 \in R\}$ is a subspace of R^3 with the standard operations (i) addition and multiplication.

Sol. The set W is nonempty subset of V because it contains the zero vector $(0,0,0)$

Let $w_1 = \{(x_1, 0, x_3)\}$

$w_2 = \{y_1, 0, y_3\}$

(i) $w_1 + w_2 = (x_1, 0, x_3) + (y_1, 0, y_3)$

$= (x_1+y_1, 0, x_3+y_3)$ by (i)

$w_1 + w_2 \in W$

(ii) $\alpha w = \alpha(x_1, 0, x_3)$

$= (\alpha x_1, 0, \alpha x_3)$ by (ii)

$\alpha w \in W$

Other 7 axioms of vector space can be verified by standard operations. Hence W is subspace of $V(R^3)$.

Exp. Show that W is a subspace of the vector space of all 2×2 symmetric matrices with standard operations of matrix addition and scalar multiplication.

(i) Closed under addition $A = A^T$ & $B = B^T$

$(A+B)^T = A^T + B^T = A+B$ So A and B are symmetric

(ii) Closed under scalar multiplication $A = A^T \Rightarrow (cA)^T = cA^T = cA$

Finite-dimensional vector space over F.

The vector space V is said to be finite dimensional over F if there is a finite subset S in V such that $V = L(S)$

$F^{(n)}$ is finite-dimensional over F, for its consists of the n vectors $(1,0,0,\dots,0), (0,1,0,\dots,0), \dots, (0,0,0,\dots,1)$; then $V = L(S)$

Thus, $L(S)$ is the set of Linear Combinations of finite number of elements of S.

Remark- $S \subseteq L(S)$

Subspace Spanned by a Set:- If S is a non-empty subset of V(F) such that $L(S) = W$, we say that the subspace W is spanned by S or S spans W

Exp. Is the vector $(3,-4,6)$ in the subspace of R^3 spanned by the vectors $(1,2,-1), (2,2,1)$ and $(1,-2,3)$?

Solution Let $W = L(S)$ be the subspace of R^3 spanned by

$$S = \{ V_1 = (1,2,-1),\ V_2 = (2,2,1),\ V_3 = (1,-2,3) \}$$

Let $V = (3,-4,6) = \alpha V_1 + \beta V_2 + \gamma V_3$ where $\alpha, \beta, \gamma \in R$

Then

$$(3,-4,6) = \alpha(1,2,-1) + \beta(2,2,1) + \gamma(1,-2,3) \quad \text{—(I)}$$
$$= (\alpha+2\beta+\gamma,\ 2\alpha+2\beta-2\gamma,\ -\alpha+\beta+3\gamma)$$
$$\alpha + 2\beta + \gamma = 3 \quad \text{—(II)}$$
$$2\alpha + 2\beta - 2\gamma = -4 \quad \text{—(III)}$$
$$-\alpha + \beta + 3\gamma = 6 \quad \text{—(IV)}$$

Solving the Eqn (II), (III) & (IV) $\alpha = 2,\ \beta = -1$ & $\gamma = 3$

Substituting the value of α, β & γ in Eqn (1)

$$(3,-4,6) = 2(1,2,-1) + (-1)(2,2,1) + 3(1,-3,3)$$
$$= (2,4,-2) + (-2,-2,-1) + (3,-6,9)$$
$$= (2-2+3,\ 4-2-6,\ -2-1+9)$$
$$= (3,-4,6)$$

Hence $v \in W = L(S)$

Exp. Show that the vector $(2,-5,3)$ is not in the subspace of R^3 spanned by the vectors $(1,-3,2)$, $(2,-4,-1)$ and $(1,-5,7)$.

Solution Let $v = (2,-5,3) \in W$

$S = \{v_1 = (1,-3,2),\ v_2 = (2,-4,-1),\ v_3 = (1,-5,7)\}$

Let $v = \alpha v_1 + \beta v_2 + \gamma v_3$

$$(2,-5,3) = \alpha(1,-3,2) + \beta(2,-4,-1) + \gamma(1,-5,7) \quad \text{—(1)}$$
$$= (\alpha,-3\alpha,2\alpha) + (2\beta,-4\beta,-\beta) + (\gamma,-5\gamma,7\gamma)$$
$$= (\alpha+2\beta+\gamma,\ -3\alpha-4\beta-5\gamma,\ 2\alpha-\beta+7\gamma)$$

$\alpha + 2\beta + \gamma = 2$ —(ii)

$-3\alpha - 4\beta - 5\gamma = -5$ —(iii)

$2\alpha - \beta + 7\gamma = 3$ —(iv)

From $3\times$(ii) + (iii), $2\beta - 2\gamma = 1 \Rightarrow \beta - \gamma = \frac{1}{2}$ —(5)

From $2\times$(ii) − (iv), $5\beta - 5\gamma = 1 \Rightarrow \beta - \gamma = \frac{1}{5}$ —(6)

Eqn (4) & (5) are inconsistance, so we cannot find values of α, β, γ clearly.

Hence the vector $(2,-5,3)$ cannot be expressed as a linear combination of the vectors $(1,-3,2)$, $(2,-4,-1)$ and $(1,-6,7)$.

Hence vector $(2,-5,3)$ is not in the subspace spanned by vectors.