Q.) In a spherical triangle ABC, show that :

$$\tan\left(\frac{a+b}{2}\right)\cot\frac{c}{2}=\frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}$$

sol :-

$$\tan\frac{A}{2}=\sqrt{\left[\frac{\sin(s-b)\sin(s-c)}{\sin s\,\sin(s-a)}\right]}$$

and $$\tan\frac{C}{2}=\left[\frac{\sin(s-a)\sin(s-b)}{\sin s\,\sin(s-c)}\right]$$

$$\therefore\ \tan\frac{A}{2}\tan\frac{C}{2}=\frac{\sin(s-b)}{\sin s}$$

Similarly $$\tan\frac{B}{2}\tan\frac{C}{2}=\frac{\sin(s-a)}{\sin s}$$

$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{\sin(s-c)}{\sin s}$$

Now $$\tan\frac{A+B}{2}\tan\frac{C}{2}=\frac{(\tan A/2+\tan B/2)}{1-\tan A/2\tan B/2}\tan\frac{C}{2}$$

$$=\frac{\frac{\sin(s-b)}{\sin s}+\frac{\sin(s-a)}{\sin s}}{1-\frac{\sin(s-c)}{\sin s}}$$

$$=\frac{\sin(s-b)+\sin(s-a)}{\sin s-\sin(s-c)}$$

or, $$\tan\frac{A+B}{2}\tan\frac{C}{2}=\frac{2\sin\frac{2s-a-b}{2}\cos\frac{a-b}{2}}{2\sin\frac{s-s+c}{2}\cos\frac{s+s-c}{2}}$$

$$=\frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\quad\left[\because 2s=a+b+c\right]$$

$$\therefore \tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{c}{2}$$

Again we know that

$$\tan\frac{a}{2} = \sqrt{\left\{\frac{-\cos S\cos(s-A)}{\cos(s-B)\cos(s-c)}\right\}}$$

$$\therefore \tan\frac{a}{2}\cot\frac{c}{2}$$

$$= \sqrt{\left\{\frac{-\cos S\cos(S-A)}{\cos(s-B)\cos(s-c)}\right\}}\sqrt{\left\{\frac{\cos(S-A)\cos(S-B)}{-\cos S\cos(S-c)}\right\}}$$

$$= \frac{\cos(S-A)}{\cos(S-c)}$$

similarly $\tan\frac{b}{2}\cot\frac{c}{2} = \dfrac{\cos(S-B)}{\cos(S-c)}$

and $\tan\frac{a}{2}\tan\frac{b}{2} = \dfrac{-\cos S}{\cos(s-c)}$

Now $\tan\frac{a+b}{2}\cot\frac{c}{2} = \dfrac{\tan\frac{a}{2}+\tan\frac{b}{2}}{1-\tan\frac{a}{2}\tan\frac{b}{2}}\cot\frac{c}{2}$

$$\frac{\frac{\cos(S-A)}{\cos(S-c)} + \frac{\cos(S-B)}{\cos(S-c)}}{1+\frac{\cos S}{\cos(S-c)}} = \frac{\cos(S-A)+\cos(S-B)}{\cos(S-c)+\cos S}$$

or, $\tan\frac{a+b}{2}\cot\frac{c}{2} = \dfrac{2\cos\frac{2S-A-B}{2}\cos\frac{A-B}{2}}{2\cos\frac{S-C-S}{2}\cos\frac{S-C+S}{2}}$

$$= \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}} \qquad \because A+B+C = 2S$$

$$\therefore \tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}$$

Similarly we can prove that

$$\tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2}$$