Q.) In a spherical triangle ABC prove that :

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$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b).\sin(s-c)}{\sin s \cdot \sin(s-a)}}$$

sol:- We know that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

Now we know that

$$2\sin^2 A/2 = 1 - \cos A$$

or, $$2\sin^2\frac{A}{2} = 1 - \frac{\cos a - \cos b\cos c}{\sin b \sin c}$$

or, $$2\sin^2\frac{A}{2} = \frac{(\cos b\cos c + \sin b\sin c) - \cos a}{\sin b\sin c}$$

or, $$2\sin^2\frac{A}{2} = \frac{\cos(b-c) - \cos a}{\sin b\sin c}$$

or, $$2\sin^2\frac{A}{2} = \frac{2\sin(b-c) - \cos a}{\sin b\sin c}$$

or, $$2\sin^2\frac{A}{2} = \frac{2\sin\frac{b-c+a}{2} \cdot \sin\frac{a-b+c}{2}}{\sin b\sin c}$$

If we take that $2s = a+b+c$

then $a+b-c = a+b+c-2c = 2(s-c)$

and $a+c-b = a+b+c-2b = 2(s-b)$;

$$\therefore \sin^2\frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b\sin c} \quad \text{—} (1)$$

$$\sin\frac{A}{2} = \sqrt{\left[\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}\right]} \quad \text{——— (1)}$$

we have taken the +ve sign with the radical as we know that A is less than two right angles i.e. $A/2$ is less than a right angle and hence its sine, Cosine and tangent are all +ve.

Again we know that

$$2\cos^2 A/2 = 1 + \cos A$$

or, $$2\cos^2\frac{A}{2} = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

or, $$2\cos^2\frac{A}{2} = \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c}$$

or, $$2\cos^2\frac{A}{2} = \frac{\cos a - \cos(b+c)}{\sin b \sin c}$$

or, $$2\cos^2\frac{A}{2} = \frac{2\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}}{\sin b \sin c}$$

or, $$\cos^2\frac{A}{2} = \frac{\sin s \cdot \sin(s-a)}{\sin b \sin c}$$

$$\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \quad \text{——— (2)}$$

From (1) and (2) we have

$$\tan\frac{A}{2} = \frac{\sin A/2}{\cos A/2} = \sqrt{\frac{\sin(s-b)\sin(s-c}{\sin s \cdot \sin(s-a}}$$