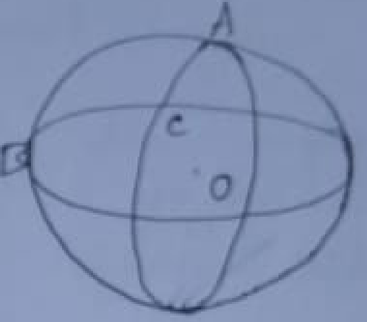
Q.) Define spherical triangle and prove that the angles of a spherical triangle can not be greater than two right angles.

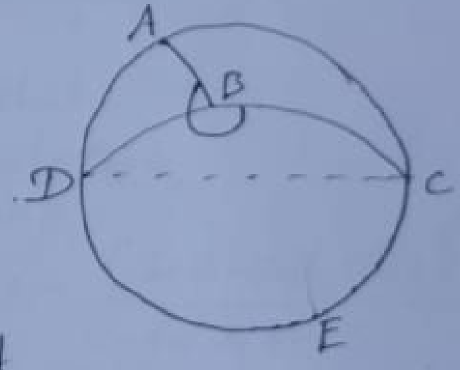
Sol:- Spherical triangles.

Let A, B and C be any three points on the surface of a sphere. We join AB, BC and CA by arcs of the great circles passing through them. The figure thus formed is called spherical triangle ABC.



The angles of a spherical triangle cannot be greater than two right angles.

Let the triangle ABc be formed by arcs AB, Bc and CEDA having angle ABC greater than two right angles.



We produce CB to meet the great circle at D. We know that two great circles bisect each other; therefore arc CED is a semi circle and as such the arc CEDA of the spherical triangle is greater than two right angles which is contrary stated above that the sides of a spherical triangle are each less than two right angles. Hence none of the angles of a spherical triangle can be greater than two right angles.

Q.) Prove that the sines of the angles of a spherical triangle are proportional to the sines of the opposite sides. 2019

i.e.

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

We note the similarity of this formula with the corresponding sine formula of plane trigonometry which states that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof :- We know that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$\therefore \sin^2 A = 1 - \cos^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$\text{or, } \sin^2 A = \frac{(1-\cos^2 b)(1-\cos^2 c) - (\cos^2 a + \cos^2 b \cos^2 c - 2\cos a \cos b \cos c)}{\sin^2 b \sin^2 c}$$

$$\text{or } \sin^2 A = \frac{(1-\cos^2 b - \cos^2 c + \cos^2 b \cos^2 c) - (\cos^2 a + \cos^2 b \cos^2 c - 2\cos a \cos b \cos c)}{\sin^2 b \sin^2 c}$$

$$\text{or, } \sin A = \frac{(1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c)^{1/2}}{\sin b \sin c}$$

We have taken only +ve sign with the radical as we know that the angles and the sides of a spherical triangle are each less than two right angles and as such $\sin A$, $\sin b$ and $\sin c$ are all +ve.

$$\therefore \frac{\sin A}{\sin a} = \frac{\left(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c\right)^{1/2}}{\sin a\sin b\sin c}$$

The symmetry of the result shows that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$= \frac{\left(1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c\right)^{1/2}}{\sin a\sin b\sin c}$$

[Note:- The expression

$$1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c = 4n^2$$

$$\therefore \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a\sin b\sin c}$$

In determinant form

$$4n^2 = \begin{vmatrix} 1 & \cos c & \cos b \\ \cos c & 1 & \cos a \\ \cos b & \cos a & 1 \end{vmatrix}$$