LINEAR PROGRAMMING

Some more definitions and example

1. A Feasible solution (A.F.S.) :- A feasible solution to a L.P. problem is the set of values of the variables which satisfies the set of Constraints and the non - negative restrictions of the problem.

2. Optimum (or optimal) solution :- A feasible solution to a L.P. problem is said to be optimum (or optimal) solution if it also optimizes the objective function Z of the problem.

3. A Basic feasible solution (A.B.F.S.) :- In a linear programming problem a feasible solution which is also basic is called a basic feasible solution (B.F.S.). In other words a feasible solution to a L.P.P. in which the vectors associated to non-zero variables (non zero variables are certainly positive in F.S.) are L.I. is called a basic feasible solution.

Since almost m vectors of E^m, Euclidean space of m- dimensions where m is the number of Constraints may be L.I. Hence a B.F.S. cannot have more than m non zero (i.e. positive) variables. Thus for a F.S. to be a B.F.S. at least

$(n-m)$ variables must vanish.

B.F.S. are also finite in number and maximum number of them is "${}^nC_m$"

4. <u>Basic variables</u>:- The non-zero variables in a B.F.S. are called basic variables.

5. <u>Non-degenerate B.F.S.</u>:- A B.F.S. of a L.P. problem is said to be non-degenerate B.F.S. if none of the basic variables is zero.

6. <u>Degenerate B.F.S.</u>:- A B.F.S. of a L.P. problem is said to be degenerate B.F.S. if at least one of the basic variables is zero.

<u>Example</u>:- Find all the basic solutions of the following system.

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_2 = 5$$

and prove that they are non-degenerate

sol:- In matrix form the given system of equations can be written as

$$Ax = b$$

where $A = (\alpha_1, \alpha_2, \alpha_3)$

$\alpha_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\alpha_2 = \begin{bmatrix}2\\1\end{bmatrix}$, $\alpha_3 = \begin{bmatrix}1\\5\end{bmatrix}$.

$x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $b = \begin{bmatrix}4\\5\end{bmatrix}$

This problem can have at most ${}^3C_2 = 3$ basic solutions.

Now the three sets of two vectors are

$B_1 = [\alpha_1, \alpha_2] = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$.

$B_2 = [\alpha_1, \alpha_3] = \begin{bmatrix}1 & 1\\2 & 5\end{bmatrix}$

$B_3 = [\alpha_2, \alpha_3] = \begin{bmatrix}2 & 1\\1 & 5\end{bmatrix}$

Here $|B_1| = -3 \neq 0$, $|B_2| = 3 \neq 0$

$|B_3| = 9 \neq 0$

since none of these is zero, therefore every set of two vectors of A are L.I. Hence all the three basic solutions exist.

If x_{B_i}, $i = 1, 2, 3$ are the vectors of the basic variables associated to the set B_i, $i = 1, 2, 3$, respectively, then

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_{B_1} = B_1^{-1} b = -\frac{1}{3}\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_3\end{bmatrix} = x_{B_2} = B_2^{-1} b = \frac{1}{3}\begin{bmatrix}5 & -1\\-2 & 1\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}$$

$$= \begin{bmatrix}5\\-1\end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = x_{B_3} = B_2^{-1} b = \frac{1}{9}\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 \\ 2/3 \end{bmatrix}$$

Hence the basic solutions are

$x_1 = (2, 1, 0)$, $x_2 = (5, 0, -1)$

$x_3 = (0, 5/3, 2/3)$

Now the determinants of matrices $[\alpha_1, b]$ and $[\alpha_2, b]$ are not zero, i.e. b and every set of $m-1 = 2-1 = 1$ Columns of B_1 are L.I. Hence by the theorem 1, the basic solution x_1 is non degenerate. Similarly we can prove that the other basis solutions x_2 and x_3 are also non-degenerate.