Abstract Algebra

SOLUTIONS OF POLYNOMIALS EQUATIONS BY RADICALS :-

Definition :- Let F be a field and an element $a \in F$ be such that it is not the n^{th} power of an element of F. So that $x^n - a \in F[x]$ has not root in F. then a root of $x^n - a$, denoted by $\sqrt[n]{a}$ is called a radical of exponent n over F.

Definition :- A radical $\sqrt[n]{a}$ of exponent n over F is known as reducible or irreducible over F according as $x^n - a$ is reducible or irreducible polynomial of $F[x]$.

Definition :- The equation of the $x^n - a = 0$ is known as pure equation.

An extension field $F(\sqrt[n]{a})$ is known as pure extension of F.

Definition :- A field extension which can be reached through a finite series of successive pure extension is referred to as solvable by radicals or as a Tower.

If K be a given field extension of a field F and we have

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_r = K$$

Such that each member of the series is a pure extension of its predecessor, then K is referred to as a Tower over F or as a solvable by radicals over F.

Solution by Radicals

Let F be a field and $f(x) \in F[x]$ be a non-Constant polynomial. In this section we shall Consider the problem of solving the equation $f(x) = 0$.

If $f(x) = x^2 + bx + c$ over a field F of characteristic $\neq 2$ then the solutions of $f(x) = 0$ are given by $-b \pm \sqrt{b^2 - 4c}$, so that this equation either has solutions in F or solutions in $F(\sqrt{b^2-4c})$. Thus $f(x) = 0$ can always be solved in some field which is obtained by adjoining a radical to F. we wish to determine when an arbitrary polynomial equation $f(x) = 0$ can be solved, if it is not solved in F, then there exists some field in which $f(x) = 0$ will have the solution, this field is obtained from F by the successive adjunction of radicals.

Definition :- Let $f(x) \in F[x]$ be a polynomial, then the polynomial equation $f(x) = 0$ is said to be solvable by radicals over F, if the splitting field K of $f(x)$ is a tower over F.

If K is a normal extension of F, then the tower is said to be normal radical tower over F.