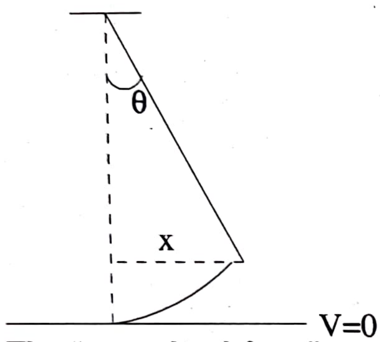
$V(q_1, q_2, \ldots, q_N)$. Lagrange defined equilibrium as a configuration in which all generalised forces vanish, i.e. $\dfrac{\partial V}{\partial q_i} = 0$. Clearly, in such a situation, the system will not change its configuration. However, even when $Q_i = 0$, the system may not be stable in the sense that if it is slightly disturbed from a position of equilibrium, it may not return to the position of equilibrium. If it does, such a configuration is called one of *stable* equilibrium - otherwise the equilibrium is unstable.

Example: Simple Pendulum



The potential energy is given by $V(\theta) = +mgl(1 - \cos\theta)$, so that

$$F(\theta) = -\frac{\partial V}{\partial \theta} = -mgl\sin\theta = -mgx$$

The "generalised force" corresponding to θ in this case is actually the restring torque. Equilibrium occurs when the restoring torque is zero. There are two such positions, $\theta = 0$ and $\theta = \pi$.

Let us look at the form of the Lagrangian near these two positions.

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

Near $\theta = 0$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ so that

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\theta^2$$

so that the potential energy is $V(\theta) = \frac{1}{2}mgl\theta^2$ and the corresponding generalised force is $-mgl\theta$ which is of restoring nature. On the other hand, near the second position of equilibrium $\theta = \pi$, $\cos\theta = \cos(\pi + \delta\theta) \approx -\cos\delta\theta = -1 + \frac{1}{2}\delta\theta^2$. In this situation,

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl(\delta\theta)^2$$

the corresponding force is "anti-restoring", making the equilibrium unstable.

For one dimensional holonomic systems, equilibrium can be either stable on unstable (leaving out a trivial case of neutral equilibrium where the potential energy function is spatially flat) for which the potential energy has an extremum

$$\frac{\partial V}{\partial q_i} = 0 \tag{1}$$

for every generalised coordinate q_i. Let the position of equilibrium be q_{i0}. If the position is one of stable equilibrium, the potential energy has to be minimum. This is because, the system being conservative, the total energy is constant. If we go away from the position of minimum potential energy, it leads to an increase in the potential energy and a consequent decrease in the kinetic energy. Thus the system returns back to the equilibrium position. For stable equilibrium, we, therefore, have

$$\frac{\partial^2 V}{\partial q_i \partial q_j} > 0 \tag{2}$$

The converse would be true for an unstable equilibrium.

Without loss of generality, let us shift the equilibrium position to the origin ($q_1 = q_2 = \ldots, = q_N = 0$). If the system is disturbed to a configuration $\{q_i\}$, we can write,

$$V(q_1, q_2, \ldots) = V(0, 0, \ldots) + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_0 q_i + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 q_i q_j + \text{ higher order terms}$$

where the partial derivatives are evaluated at the position of equilibrium and all "higher order terms" which involve third order or higher corrections are neglected. If the potential energy is measured from its minimum value, we choose $V(0, 0, \ldots) = 0$. Along with

$$\left(\frac{\partial V}{\partial q_i}\right)_0 = 0$$

The leading term in the change in potential energy is then

$$\frac{1}{2}\sum_{i,j}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 > 0$$

for stable equilibrium. Let us write

$$V_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0$$

so that

$$V = \sum_{i,j} \frac{1}{2} V_{ij} q_i q_j \tag{3}$$

with $V_{ij} = V_{ji}$.

Now, the kinetic energy of a scleronomic system is, in general, a quadratic in generalised velocities, and can be written as

$$T = \frac{1}{2}\sum_{i,j} T_{ij} \dot{q}_i \dot{q}_j \tag{4}$$

where the coefficients t_{ij} are, in general, functions of generalised coordinates. One can expand t_{ij} in a Taylor series about the equilibrium position

$$T_{ij}(q_1, q_2, \ldots) = t_{ij}(0, 0, \ldots) + \sum_k \left(\frac{\partial t_{ij}}{\partial q_k}\right)_0 q_k + \ldots$$

It turns out that the quantities $\left(\frac{\partial t_{ij}}{\partial q_k}\right)_0$ and the higher order derivatives are negligibly small so that the coefficients t_{ij} s can be essentially treated as constants having the same values as they would have in the equilibrium position. Thus around the equilibrium position, the Lagrangian has the following structure:

$$\mathcal{L} = T - V = \frac{1}{2}\sum_{i,j}(t_{ij}\dot{q}_i\dot{q}_j - V_{ij}q_iq_j)$$

The Lagrangian equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

can then be written as

$$\frac{d}{dt}\sum_{i,j}\frac{1}{2}\left[t_{ij}\delta_{ik}\dot{q}_j + t_{ij}\dot{q}_i\delta_{kj}\right] - \frac{1}{2}\sum_{i,j}V_{ij}\left(\delta_{ik}q_j + q_i\delta_{kj}\right) = 0$$

$$\frac{1}{2}\left[\sum_j t_{kj}\ddot{q}_j + \sum_i t_{ik}\ddot{q}_i\right] - \frac{1}{2}\left[\sum_j V_{ik}q_j + \sum_i V_{ik}q_i\right] = 0 \qquad (5)$$

Changing the dummy summation index j to i in the first and the third terms of the above and using the symmetry of V_{ij} and of t_{ij}, we get

$$\sum_i t_{ik}\ddot{q}_i + \sum_i V_{ik}q_i = 0$$

for each k. We seek a solution to the above equations of the form

$$q_i = A_i e^{i\omega t}$$

which gives

$$\sum_i \left(V_{ik} - \omega^2 t_{ik}\right) A_i = 0 \qquad (6)$$

The equation is a homogeneous equation in A_is and the condition for existence of the solution is

$$\det(V_{ik} - \omega^2 t_{ik}) = 0$$

which is a single algebraic equation of $n-$ th degree in ω^2. This equation has n roots some of which are real and some complex (some of the roots may be degenerate). We are

only interested in real roots of the above equation. ω_k's determined from this equation are known as **characteristic feequencies** or **eigenfrequencioes.**
From physical arguments it is clear that for real physical situations, the roots are real and positive. This is because the existence of an imaginary part in ω would mean time dependence of q_k and $\dot{q}_k$ such that the total energy would not be conserved in time and such solutions are unacceptable.
We can arrive at the same conclusion mathematically as well. Multiplying (6) with A_k^* and summing over k we get

$$\sum_{i,k}(V_{ik} - \omega^2 t_{ik})A_k^* A_i = 0$$

so that

$$\omega^2 = \frac{\sum_{i,k} V_{ik} A_k^* A_i}{\sum_{i,k} t_{ik} A_k^* A_i}$$

Both the numerator and the denominator are real because $V_{ik} = V_{ki}$ and $t_{ik} = t_{ki}$. It is seen that the terms are positive as well because expressing $A_i = a_i + ib_i$, we have

$$\sum_{i,k} t_{ik} A_i^* A_k = \sum_{i,k} t_{ik}(a_i - ib_i)(a_k + ib_k)$$
$$= \sum_{i,k} t_{ik}(a_i a_k + b_i b_k)$$

where the imaginary terms cancel because of symmetry of t_{ik}. Thus we have been able to express $\sum_{i,k} t_{ik} A_i^* A_k$ as a sum of two positive semi-definite terms ($\sum t_{ik} a_i a_k = a^T t a$ is positive definite).

2.2 Matrix Formulation

Let us rewrite (6) (using symmetry properties of V and T) as

$$\sum_i (V_{ki} - \lambda t_{ki})A_i = 0$$

where $\lambda = \omega^2$. Let us define a column vector

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \dots \\ A_N \end{pmatrix}$$

The matrices V and t are given by

$$V = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1N} \\ V_{21} & V_{22} & \dots & V_{2N} \\ \dots & \dots & \dots & \dots \\ V_{N1} & V_{N2} & \dots & V_{NN} \end{pmatrix} \qquad T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1N} \\ t_{21} & t_{22} & \dots & t_{2N} \\ \dots & \dots & \dots & \dots \\ t_{N1} & t_{N2} & \dots & t_{NN} \end{pmatrix}$$