Abstract Algebra

SIMPLE AND SEMI SIMPLE MODULES

Throughout this section all ring will be rings with unity and all R-modules will be unital. Unless otherwise specified "R-module" means left R-module.

Definition 1. An R-module M is simple if it is non-zero and has no submodules other than itself and the zero submodule.

Example 1 An abelian group $G \neq 0$ is a simple Z-module if and only if it cyclic of prime order.

Example 2 A vector space over a division ring R is a simple R-module if and only if it has dimension 1.

Example 3 If M is the left R-module formed by the additive group of a ring R. then a submodule $N \neq 0$ of M is simple if and only if it is a minimal left ideal of M.

Schur's Lemma

Let M be a simple R-module and N any R-module. Then

(i) Every non-zero homomorphism $f: M \to N$ is injective (monomorphism)

(ii) Every non-zero homomorphism $f: M \to N$ is surjective (epimorphism)

(iii) $End_R(M)$ is a division ring. where $End_R(M) = Hom_R(M, M)$.

Proof :- (i) since $f \neq 0$ is a homomorphism, so $Ker(f)$ is a submodule of M. As M is simple, therefore either $Ker(f) = \{0\}$ or $Ker(f) = M$.

But $f \neq 0$ so $Ker(f) \neq M$. Then $Ker(f) = \{0\}$. hence f is injective.

(ii) let f is a non-zero homomorphism from M to N, so $Im(f)$ is also a non-zero submodule of M and M is simple therefore $Im(f) = M$. hence f is surjective.

(iii) let f be a non-zero R-module homomorphism from M to M. Since M is simple, then by (i) and (ii), f is an automorphism. Hence f is a unit in the ring $End_R(M)$.

Consequently $End_R(M)$ is a division ring.

Definition 2

A homomorphism of a ring R into the ring $End_R(M)$ of all endomorphism of an abelian group M is called a representation R.

If M is a left R-module, then a mapping $\rho : R \to End_R(M)$, $\rho \to I_\rho$, is the representation of R associated with M, where for each $r \in R$, the mapping $I_r : M \to M$ given by $I_r(x) = rx$ is an endomorphism of M.