Q. A sequence $\{a_n\}$ is defined by $a_{n+1} = \sqrt{k + a_n}$, where $k > 0$, $a_1 > 0$. Prove that the sequence $\{a_n\}$ is monotonic & converges to the positive root of the equation $x^2 - x - k = 0$

We have $a_{n+1} = \sqrt{k + a_n}$

or, $a_{n+1}^2 = k + a_n$

putting $n = n-1$, we get

$$a_n^2 = k + a_{n-1}$$

$$\therefore a_{n+1}^2 - a_n^2 = a_n - a_{n-1}$$

This shows that
$$\left.\begin{array}{l} a_{n+1} > a_n \text{ according as } a_n > a_{n-1} \\ a_{n+1} < a_n \text{ according as } a_n < a_{n-1} \end{array}\right\} \text{— (1)}$$

Now the following cases arise:

Case I Let $a_2 > a_1$.

Then from (1) we find that $\{a_n\}$ is monotonic increasing

Also we shall prove that $\{a_n\}$ is bounded above

The given equation is $x^2 - x - k = 0$

Since the product of the roots $= -k$ and $k > 0$, therefore one root of the equation must be positive & the other negative.

Let the roots be α & $-\beta$ where $\alpha > 0$, $\beta > 0$

Then $x^2 - x - k = (x - \alpha)[x - (-\beta)] = (x - \alpha)(x + \beta)$

$\because a_{n+1}^2 = k + a_n$ and $a_{n+1}^2 > a_n^2$

$\therefore k + a_n > a_n^2$

or, $a_n^2 - a_n - k < 0$

or, $(a_n - \alpha)(a_n + \beta) < 0$ [one root is $-\beta$]

or, $a_n < \alpha \ \forall n$

Therefore $\{a_n\}$ is bounded above & hence $\{a_n\}$ is convergent.

Case II Let $a_2 < a_1$

Then from (1) we find that $\{a_n\}$ is monotonic decreasing and as in case I we can show that $\{a_n\}$ is bounded below.

Hence $\{a_n\}$ is convergent

Thus in both the cases $\{a_n\}$ tends to a limit.

Let $\lim\limits_{n \to \infty} a_n = l = \lim\limits_{n \to \infty} a_{n+1}$

We have $a_{n+1}^2 = k + a_n$ or $\lim\limits_{n \to \infty} a_{n+1}^2 = k + \lim\limits_{n \to \infty} a_n$

or, $l^2 = k + l$ or $l^2 - l - k = 0$

Hence l is a root of eq^n $x^2 - x - k = 0$

We have $a_n > 0 \ \forall n$

Therefore l can not be negative. Hence l is the positive root of the eq^n $x^2 - x - k = 0$ i.e; $\{a_n\}$ converges to the positive root of $x^2 - x - k = 0$

Q. If the sequence $\{u_n\}$ is defined by $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n}$, prove that $\frac{1}{2} < u_n < 1$ & that u_n tends to a limit as $n \to \infty$

We have $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n-2} + \frac{1}{n+n-1} + \frac{1}{n+n}$

$\therefore u_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$

$\therefore u_{n+1} - u_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$

$= \frac{1}{(2n+1)(2n+2)} > 0 \ \forall n \in N$

$\therefore u_{n+1} > u_n \ \forall n$

Hence $\{u_n\}$ is monotonic increasing

Again, $u_n < \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$

or, $u_n < n \frac{1}{n}$

or, $u_n < 1 \ \forall n$

Hence $\{u_n\}$ is bounded above.

Thus $\{u_n\}$ is convergent.

Also, $u_n > \frac{1}{n+n} + \frac{1}{n+n} + \cdots + \frac{1}{n+n}$

or, $u_n > n . \frac{1}{n+n}$

or, $u_n > \frac{1}{2}$

Thus $\frac{1}{2} < u_n < 1$

$\therefore \frac{1}{2} < \lim_{n \to \infty} u_n < 1$

i.e. $\{u_n\}$ tends to a limit as $n \to \infty$.