Q. Prove that every convergent sequence of real numbers is a Cauchy sequence

Let $\{a_n\}$ be a convergent sequence of real numbers which converges to $l \in R$

Then for a given arbitrary small positive number ϵ there exists $\mu \in N$ such that

$$n \geq \mu \Rightarrow |a_n - l| < \frac{\epsilon}{2}$$

Therefore if $m \geq \mu \Rightarrow |a_m - l| < \frac{\epsilon}{2}$

Now,
$$|a_n - a_m| = |(a_n - l) - (a_m - l)|$$
$$\leq |a_n - l| + |a_m - l|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \; (= \epsilon)$$

i.e. for a given arbitrary small positive number ϵ there exists μ such that $n, m \geq \mu \Longrightarrow |a_n - a_m| < \epsilon$

Hence by definition, $\{a_n\}$ is a Cauchy sequence.

Q. Prove that every Cauchy sequence of real numbers is bounded

Let $\{a_n\}$ be a Cauchy sequence. Then we have to prove that it is bounded.

By definition of Cauchy sequence, for a given arbitrary small positive number ϵ there exists $V \in N$ such that

$$n, m \geq V \Longrightarrow |a_n - a_m| < \epsilon$$

If we take a particular value K for m greater than V,

$$|a_n - a_K| < \epsilon \text{ for } n \geq V$$

i.e. $a_K - \epsilon < a_n < a_K + \epsilon$ for $n \geq V$

Let $\lambda = \max.\{a_1, a_2, a_3, ----, a_{n-1}, a_K + \epsilon\}$

& $\mu = \min\{a_1, a_2, a_3, ---, a_{n-1}, a_K - \epsilon\}$

Then $\mu \leq a_n \leq \lambda \;\forall n \in N$

$\therefore \{a_n\}$ is bounded.

Thus every Cauchy sequence of real numbers is bounded.

Q. If $\{u_n\}$ & $\{v_n\}$ be two convergent sequence such that $\lim_{n\to\infty} u_n = l$ & $\lim_{n\to\infty} v_n = l'$, then prove that $\{u_n + v_n\}$ is also convergent & converges to $l + l'$

Since $u_n \to l$ & $v_n \to l'$ as $n \to \infty$, therefore for a given arbitrary small positive number ϵ there exist two positive integers m_1 & m_2 such that

$$|u_n - l| < \frac{\epsilon}{2} \quad \forall\, n \geq m_1 \quad \text{--- (1)}$$

$$\& \ |v_n - l'| < \frac{\epsilon}{2} \quad \forall\, n \geq m_2 \quad \text{--- (2)}$$

Let m be the maximum of m_1 & m_2

Then (1) & (2) both hold for $n \geq m$

Now, $|(u_n + v_n) - (l + l')| = |(u_n - l) + (v_n - l')|$

$$\leq |u_n - l| + |v_n - l'|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \ (= \epsilon) \ \forall\, n \geq m \quad [\text{from (1) \& (2)}]$$

i.e. $|(u_n + v_n) - (l + l')| < \epsilon \ \forall\, n \geq m$

Hence $(u_n + v_n) \to l + l'$ as $n \to \infty$ i.e. $\{u_n + v_n\}$ is convergent.

Q. Show that the sequence $\{u_n\}$ where $u_n = n^{\frac{1}{n}}$ is convergent

If $u_n = n^{\frac{1}{n}}$, let us write $u_n = 1 + b_n$, where $b_n > 0$

Now, $(u_n)^n = n$ i.e. $(1 + b_n)^n = n$

or, $n = 1 + n b_n + \frac{n(n-1)}{\lfloor 2} b_n^2 + \cdots + b_n^n$

or, $n > \frac{n(n-1)}{\lfloor 2} b_n^2$ or $\frac{2}{n-1} > b_n^2$ or, $\sqrt{\frac{2}{n-1}} > b_n$

Therefore $0 < b_n < \sqrt{\frac{2}{n-1}}$

or, $0 < \lim_{n\to\infty} b_n < \lim_{n\to\infty} \sqrt{\frac{2}{n-1}}$

or, $0 < \lim_{n\to\infty} b_n < 0$

$\therefore \lim_{n\to\infty} b_n = 0$

Hence $\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1 + b_n) = 1$

This show that $\{u_n\}$ is convergent.