Q. Prove that the sequence $\{u_n\}$, where $u_n = \left(1+\frac{1}{n}\right)^n$ is convergent. Also show that $\{u_n\}$ converges to a limit lying between 2 & 3.

We have $u_n = \left(1+\frac{1}{n}\right)^n$

Expanding R.H.S. by Binomial ~~thereo~~ theorem, we get

$$u_n = 1 + n\frac{1}{n} + \frac{n(n-1)}{\underline{|2}}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{\underline{|3}}\frac{1}{n^3} + \cdots\cdots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{\underline{|2}}\left(1-\frac{1}{n}\right) + \frac{1}{\underline{|3}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots + \frac{1}{\underline{|n}}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{n-1}{n}\right)$$

Changing n into $n+1$, we get

$$u_{n+1} = 1 + 1 + \frac{1}{\underline{|2}}\left(1-\frac{1}{n+1}\right) + \frac{1}{\underline{|3}}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots\cdots + \frac{1}{\underline{|n}}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\cdots$$
$$\cdots\cdots\left(1-\frac{n-1}{n+1}\right) + \frac{1}{(n+1)^{n+1}}$$

In u_n & u_{n+1} we find that the first two terms are equal and the onwards terms of u_{n+1} are each greater than the corresponding terms of u_n and at the same time u_{n+1} contains an additional positive term $\frac{1}{(n+1)^{n+1}}$

$\therefore u_{n+1} > u_n$ for $n = 1, 2, 3, \cdots\cdots$

Hence the sequence $\{u_n\}$ is monotonic increasing

Now $u_n < 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots + \frac{1}{\underline{|n}}$

$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\cdots + \frac{1}{2^{n-1}}$

i.e. $u_n < 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$ i.e. $u_n < 1 + 2\left(1-\frac{1}{2^n}\right) < 1+2$ i.e. 3

i.e. $u_n < 3$

So the sequence $\{u_n\}$ is bounded above

Hence $\{u_n\}$ is convergent

Since $u_n < 3$, therefore $\lim\limits_{n\to\infty} u_n < 3$

Again $u_n > 2$, therefore $\lim\limits_{n\to\infty} u_n > 2$

Thus $2 < \lim\limits_{n\to\infty} u_n < 3$

Hence $\{u_n\}$ converges to a limit lying between 2 & 3.

Q. Show that the sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots\ldots$ Converges to the positive root of $x^2-x-7=0$

Let $\{a_n\} = \{\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots\}$

Let $a_1 = \sqrt{7}$

Then $a_2 = \sqrt{7+\sqrt{7}} = \sqrt{7+a_1}$

$a_3 = \sqrt{7+\sqrt{7+\sqrt{7}}} = \sqrt{7+a_2}$

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lly $a_n = \sqrt{7+a_{n-1}}$

$a_{n+1} = \sqrt{7+a_n}$

We see that $a_2 > a_1, a_3 > a_2, \ldots, a_n > a_{n-1}, a_{n+1} > a_n$

Hence the sequence $\{a_n\}$ is monotonic increasing

Now $a_1 = \sqrt{7}$ $\therefore a_1 < 7$

Let $a_n < 7$

$\therefore 7 + a_n < 7+7$

or $\sqrt{7+a_n} < \sqrt{14} < \sqrt{49}$

or $\sqrt{7+a_n} < 7$

or $a_{n+1} < 7$

Thus $a_n < 7 \ \forall n$

Hence $\{a_n\}$ is bounded above. So the sequence $\{a_n\}$ must converge.

Let $\lim\limits_{n\to\infty} a_n = l = \lim\limits_{n\to\infty} a_{n+1}$

Now $a_{n+1} = \sqrt{7+a_n}$ or $a_{n+1}^2 = 7 + a_n$

or, $\lim\limits_{n\to\infty} a_{n+1}^2 = \lim\limits_{n\to\infty}(7+a_n) = 7 + \lim\limits_{n\to\infty} a_n$

or, $l^2 = 7 + l$ or $l^2 - l - 7 = 0$ or $l = \dfrac{1 \pm \sqrt{1+28}}{2} = \dfrac{1\pm\sqrt{29}}{2}$

Since $\dfrac{1-\sqrt{29}}{2} < 0$, therefore $l \neq \dfrac{1-\sqrt{29}}{2}$ as $a_n > 0$

$\therefore l = \dfrac{1+\sqrt{29}}{2}$

Thus $\{a_n\}$ converges to $l = \dfrac{1+\sqrt{29}}{2}$ which is the positive root of the equation $x^2-x-7=0$