Q. State & prove Cauchy's general principle of convergence of a sequence

Statement:- The necessary & sufficient condition for the convergence of the sequence $\{a_n\}$ is that for a given arbitrary small positive number ϵ there exists a positive integer v such that

$$|a_{n+p} - a_n| < \epsilon \ \forall\ n \geq v \ \&\ \text{for all positive integral value of } p.$$

Proof:- Necessity

Let the sequence $\{a_n\}$ be convergent. Then it must tend to a finite limit. Let l be such limit. Therefore for a given arbitrary small positive number ϵ there exists a positive integer v such that

$$|a_n - l| < \frac{\epsilon}{2} \ \forall\ n \geq v \quad \text{——— (1)}$$

Then it follows that

$$|a_{n+p} - l| < \frac{\epsilon}{2} \ \forall\ n \geq v \ \&\ \text{for all } p$$

Now
$$|a_{n+p} - a_n| = |(a_{n+p} - l) - (a_n - l)|$$
$$\leq |a_{n+p} - l| + |a_n - l|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \ \forall\ n \geq v \ \&\ \text{for all } p \ [\text{from (1) \& (2)}]$$

i.e. $|a_{n+p} - a_n| < \epsilon \ \forall\ n \geq v \ \&$ for all p

Sufficiency

Let $|a_{n+p} - a_n| < \epsilon \ \forall\ n \geq v \ \&$ for all p

i.e. $a_n - \epsilon < a_{n+p} < a_n + \epsilon \ \forall\ n \geq v \ \&$ for all p

Therefore $\{a_{n+p}\}$ is bounded.

Let M & m be the l.u.b. & the g.l.b. of $\{a_{n+p}\}$ respectively.

Then $a_{n+p} > M - \epsilon$ and $a_{n+p} < m + \epsilon$ — (3)

i.e. $M < a_{n+p} + \epsilon$ and $m > a_{n+p} - \epsilon$

$\therefore M - m < (a_{n+p} + \epsilon) - (a_{n+p} - \epsilon)$

i.e. $M - m < 2\epsilon$

But since ϵ is a very small positive number, therefore $M = m$

$\therefore$ from (3) it follows that

$$M - \epsilon < a_{n+p} < M + \epsilon$$

i.e. $|a_{n+p} - M| < \epsilon$

i.e. $a_{n+p} \to M$ as $n+p \to \infty$

Hence the sequence $\{a_{n+p}\}$ is convergent & consequently the sequence $\{a_n\}$ is convergent.

Cauchy Sequence :-

A sequence $\{a_n\}$ is called a Cauchy sequence if for a given arbitrary small positive number ϵ there exists $V \in N$ such that $n, m \geq V \Longrightarrow |a_n - a_m| < \epsilon$

Q. Prove that every Cauchy sequence of real numbers is a convergent sequence

Let $\{a_n\}$ be a Cauchy sequence of real numbers. Therefore for a given arbitrary small positive number ϵ there exists $V \in N$ such that

$$n, m \geq V \Rightarrow |a_n - a_m| < \epsilon$$

Taking $m = n+p$, we get

$n \geq V \Rightarrow |a_n - a_{n+p}| < \epsilon$, p being any positive integer

i.e. $n \geq V \Rightarrow |a_{n+p} - a_n| < \epsilon$

Hence by Cauchy's general principle of convergence $\{a_n\}$ is convergent.