Q. Prove that every convergent Sequence is bounded. Also show that the converse of the theorem is not true.

Let $\{a_n\}$ be a convergent Sequence. So a_n must tend to a limit l as $n \to \infty$. Then by definition of the limit of a Sequence it follows that for a given arbitrary small positive number ϵ there exists a positive integer p such that

$$|a_n - l| < \epsilon \ \forall \ n \geq p$$

i.e. $l - \epsilon < a_n < l + \epsilon \ \forall \ n \geq p$ —— (1)

Let $M = \max.\{a_1, a_2, a_3, \ldots, a_{n-1}, l+\epsilon\}$

& $m = \min.\{a_1, a_2, a_3, \ldots, a_{n-1}, l-\epsilon\}$

Then $m \leq a_n \leq M \ \forall \ n$

Hence by definition the Sequence $\{a_n\}$ is bounded.

The converse of the theorem need not be true.

For example, let us consider the Sequence $\{a_n\}$, where $a_n = (-1)^n$

We see that the sequence $\{a_n\}$ is bounded for

$a_n = -1$ or $1 \ \forall \ n \in N$

But $\{a_n\}$ has no limit. Hence the Sequence $\{a_n\}$ is not convergent.

Q. Prove that a convergent Sequence possesses its limit uniquely.

If possible, let the convergent Sequence $\{a_n\}$ has two distinct limits l & l'.

Since $\{a_n\}$ is convergent, therefore by definition it follows that for a given arbitrary small positive number ϵ there exists a positive integer m such that

$|a_n - l| < \frac{\epsilon}{2} \quad \forall\, n \geq m$ —— (1)

& $|a_n - l'| < \frac{\epsilon}{2} \quad \forall\, n \geq m$ —— (2)

Now, $|l - l'| = |(a_n - l') - (a_n - l)|$

$\leq |a_n - l'| + |a_n - l|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall\, n \geq m$ [from (1) & (2)]

i.e. $|l - l'| < \epsilon \quad \forall\, n \geq m$

But ϵ is an arbitrary small positive number, therefore $l = l'$.
Hence the limit of the sequence is unique.

Q. Prove that every monotonic increasing sequence tends to its least upper bound.

Let $\{a_n\}$ be a monotonic increasing sequence whose least upper bound is M.

Then by definition of upper bound,

$a_n \leq M \quad \forall\, n$ —— (1)

& $a_n > M - \epsilon$ for at least one value of n, where ϵ is an arbitrary small positive number.

Let this be true for $n = m$

Then $a_m > M - \epsilon$ —— (2)

Since the sequence $\{a_n\}$ is monotonic increasing, therefore

$a_n \geq a_m$ for all $n \geq m$ —— (3)

From (2) & (3), we get

$a_n > M - \epsilon \quad \forall\, n \geq m$ —— (4)

Again from (1), $a_n \leq M \quad \forall\, n$

$\therefore a_n < M + \epsilon \quad \forall\, n$ & hence also for $n \geq m$ —— (5)

Combining the results (4) & (5), we have

$M - \epsilon < a_n < M + \epsilon \quad \forall\, n \geq m$

i.e. $|a_n - M| < \epsilon \quad \forall\, n \geq m$

Hence $\{a_n\}$ tends to M, the least upper bound of $\{a_n\}$
i.e. every monotonic increasing sequence tends to its l.u.b.