Sequences & their limits

Bounded Sequence: If there exists a finite number M such that $u_n \leq M \ \forall \ n \in N$, then the sequence $\{u_n\}$ is said to be bounded above.

If there exists a number m such that $u_n \geq m \ \forall \ n \in N$, then the sequence $\{u_n\}$ is said to be bounded below.

A sequence $\{u_n\}$ which is bounded above and bounded below is said to be bounded sequence.

M & m are respectively called the rough upper bound and rough lower bound of the sequence.

Note:- A sequence of elements $u_1, u_2, u_3, \text{----}, u_n, \ldots$ is denoted by $\{u_n\}$ or $\langle u_n \rangle$

Least upper bound:

The least among the upper bounds of the sequence is called the least upper bound (l.u.b.) of the sequence.

Alter: If there exists a number M such that

(i) $u_n \leq M \ \forall \ n \in N$

(ii) $u_n > M - \epsilon$ for at least one value of n, where ϵ is an arbitrary small positive number, then M is called the l.u.b. of $\{u_n\}$

Greatest lower bound:

The greatest among the lower bounds of the sequence is called the greatest lower bound (g.l.b.) of the sequence.

Alter:

If there exists a number m such that

(i) $u_n \geq m \ \forall \ n \in N$

(ii) $u_n < m + \epsilon$ for at least one value of n, then m is called the g.l.b. of $\{u_n\}$

Ex: In the sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \text{---}, \frac{n}{n+1}\right\}$ the l.u.b. is 1 and the g.l.b is $\frac{1}{2}$

Monotonic increasing Sequence & Monotonic decreasing Sequence:

If there exists a Sequence $\{u_n\}$ such that $u_{n+1} \geq u_n \ \forall n \in N$, then $\{u_n\}$ is called monotonic increasing Sequence.

If there exists a Sequence $\{u_n\}$ such that $u_{n+1} \leq u_n \ \forall n \in N$, then $\{u_n\}$ is called monotonic decreasing Sequence.

A Sequence which is either monotonic increasing or monotonic decreasing is said to be monotonic.

Ex. The Sequence $\{1, 2, 3, \ldots, n\}$ is monotonic increasing & the Sequence $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\}$ is monotonic decreasing.

Limit of a Sequence:

Let $\{u_n\}$ be a sequence. If corresponding to any arbitrary small positive number ϵ there exists a positive integer m depending on ϵ such that $|u_n - l| < \epsilon \ \forall n \geq m$

i.e. $l - \epsilon < u_n < l + \epsilon \ \forall n \geq m$, where l is a finite number, then l is called the limit of $\{u_n\}$ & this can be written as $\lim_{n\to\infty} u_n = l$ or $u_n \to l$ as $n \to \infty$.

Convergent Sequence:

If there exists a Sequence $\{u_n\}$ such that $u_n \to l$ as $n \to \infty$ (l is a finite number), then $\{u_n\}$ is said to be convergent.

Ex: the Sequence $\left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots, \frac{2n-1}{2n}\right\}$ is convergent because here $\lim_{n\to\infty} u_n = 1$ (finite) $\left[u_n = \frac{2n-1}{2n}\right]$

Divergent Sequence:

If there exists a Sequence $\{u_n\}$ such that $u_n \to -\infty$ or ∞ as $n \to \infty$, then $\{u_n\}$ is said to be divergent.

Ex. The Sequence $\{1, 2, 3, \ldots, n\}$ is divergent because here $\lim_{n\to\infty} u_n = \infty$