Abstract Algebra

Separable and Inseperable extensions

Definition :- An irreducible polynomial $f(x) \in F[x]$ is said to be seperable over F, if $f(x)$ has no multiple roots in its splitting field i.e. the roots of $f(x)$ in its splitting field are all simple.

A polynomial which is not seperable, is called inseparable.

Notes :-

(i) An irreducible polynomial is seperable iff $f'(x) \neq 0$.

(ii) Every non-zero polynomial over a field of characteristic zero is seperable.

Definition :- Let K be an algebraic extension of a field F. Then an element $\alpha \in K$ is said to be seperable element over F if the minimal polynomial for α over F is seperable otherwise α is inseperable element over F.

Definition :- An algebraic extension K of a field F is said to be seperable extension of F if every element of K is seperable over F. Otherwise K is said to be inseperable extension.

Notes :- If F is a field of characteristic zero, then any algebraic extension of F is seperable.

Definition :- A field F is said to be perfect if each of its extension is seperable.

PRIMITIVE ELEMENT

Definition Let K be an extension of a field F, then an element $\alpha \in K$ is said to be primitive element if
$K = F(\alpha)$

Theorem
(Primitive element theorem):- Any finite extension of a field F of characteristic zero is simple.

Proof :- Since every finite extension is algebraic. Let F be a field of characteristic zero. Let K be any finite extension of F and let $[K:F] = n$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for K over F, then

$$K = F(\alpha_1, \alpha_2, \ldots, \alpha_n) = (F(\alpha_1, \alpha_2))(\alpha_3, \alpha_n, \ldots, \alpha_{n-1}, \alpha_n)$$
$$= (F(\gamma_1))(\alpha_3, \alpha_n, \ldots, \alpha_{n-1}, \alpha_n)$$
$$[F(\gamma_1) = F(\alpha_1, \alpha_2)]$$

$$= (F(\gamma_1, \alpha_3))(\alpha_n, \ldots, \alpha_{n-1}, \alpha_n)$$
$$= F(\gamma_2)(\alpha_n, \ldots, \alpha_{n-1}, \alpha_n)$$

[Again by above theorem $F(\gamma_1, \alpha_3) = F(\gamma_2)$]
Continue to the repeated use of above theorem, we arrive that
$$K = F(\gamma_{n-2}, \alpha_n) = F(\gamma_{n-1})$$
$$[\because F(\gamma_{n-1}) = F(\gamma_{n-2}, \alpha_n)]$$

This show that K is a simple extension. Since K is an arbitrary element finite extension of F. Hence every finite extension of F of characteristic zero is simple extension.