IInd Comparison Test : If $\sum a_n$ and $\sum b_n$ are two positive term series such that

$$\frac{a_n}{a_{n+1}} \geq \frac{b_n}{b_{n+1}} \quad n \geq m. \quad \text{---} (1)$$

then (i) $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

(ii) $\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges

Proof: for $n \geq m$ we have

$$\frac{a_m}{a_n} = \frac{a_m}{a_{m+1}} \cdot \frac{a_{m+1}}{a_{m+2}} \cdot \frac{a_{m+2}}{a_{m+3}} \cdots \frac{a_{n-1}}{a_n}$$

from (1) $\frac{a_n}{a_{n+1}} \geq \frac{b_n}{b_{n+1}}$

$$\frac{a_m}{a_n} \geq \frac{b_m}{b_{m+1}} \cdot \frac{b_{m+1}}{b_{m+2}} \cdot \frac{b_{m+2}}{b_{m+3}} \cdots \frac{b_{n-1}}{b_n}$$

$$\frac{a_m}{a_n} = \frac{b_m}{b_n}.$$

Thus $\frac{a_m}{a_n} \geq \frac{b_m}{b_n}$ or $\frac{a_n}{a_m} \leq \frac{b_n}{b_m}$ for $n \geq m$

$\therefore$ $a_n \leq \frac{a_m}{b_m} \cdot b_n$ or $a_n \leq k b_n$ $\forall n \geq m$ — (2)

where $k = \frac{a_m}{b_m}$ is a fixed positive number

Applying I^{st} comparition Test in (2) we obtain.

(i) $\sum b_n$ converges $\Rightarrow \sum a_n$ converges

(ii) $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges

As I^{st} Comparison Test

let $\sum a_n$ and $\sum b_n$ be two positive term series such that

$$a_n \leq k\, b_n \quad \forall\ n \geq m \quad \text{——— (III)}$$

k is a fixed positive number and m is a fixed positive integer.

then (i) $\sum b_n$ converges $\Rightarrow \sum a_n$ converges

(ii) $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges.

proof in preceding lecture.

Such as $S_n \leq kT_n + a$ where $a = S_m - kT_m$ — (4)

S_n and T_n be the sequences of partial sums of the positive term series $\sum a_n$ and $\sum b_n$, respectively.

(i) Suppose $\sum b_n$ converges

By a fundamental test for positive term series $\sum a_n$ converges $\Leftrightarrow S_n < k \ \forall n$.

Then the sequence $\langle T_n \rangle$ of partial sum of $\sum b_n$ is bounded above. i.e there exists a positive number t such that

$$T_n \leq t \quad \forall n. \quad \text{—⑤}$$

from ④ & ⑤

$$S_n \leq kt + a \quad \forall n$$

Thus the sequence $\langle S_n \rangle$ of partial sum of $\sum a_n$ is bounded above and so $\sum a_n$ is convergent.

(ii) Suppose $\sum a_n$ diverges

Then $\lim_{n \to \infty} S_n \to +\infty$ —⑥

from ④ & ⑥, we obtain

$$T_n \geq \frac{1}{k}(S_n - a) \Rightarrow \lim_{n \to \infty} T_n \to +\infty$$

$\because k > 0$

It follows that the sequence $\langle T_n \rangle$ of partial sum of the series $\sum b_n$ diverg and so $\sum b_n$ diverges.