

Q. State and prove Hamilton's principle. —

Ans:- Let us consider the three-dimensional motion of a particle of mass m in a central potential. In Cartesian coordinates the kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and the potential energy $V(x, y, z)$ depends only on the magnitude of $r = (x^2 + y^2 + z^2)^{1/2}$.

We introduce the Lagrangian

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \quad \text{--- (1)}$$

It has derivatives

$$\frac{\partial L}{\partial x} = \frac{\partial V}{\partial x} = F_x$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = P_x$$

With similar expressions for the y and z components. The equation of motion, $F_x = \dot{P}_x$ can thus be written

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0$$

which has the form of an Euler-Lagrange equation for the integral

$$I = \int_{t_0}^{t_1} L dt. \quad \text{--- (2)}$$

This integral is called the action and Hamilton's principle of least action states that a system will evolve in such a way as to minimize the action.

The above formalism is readily extended to the case where the Lagrangian can be expressed in terms of generalized coordinates $q_i(t)$ and their time derivatives $\dot{q}_i(t)$. Then, the Euler-Lagrange equations become

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad (i = 1, 2, 3) \quad \text{--- (3)}$$

They ensure that the action is stationary with respect to arbitrary variations δq_1 , δq_2 and δq_3 subject to the condition that the variations are zero at the limits of integration t_0 and t_1 .

Lagrange's equations are applicable not only to classical and quantum particle dynamics but also to classical and relativistic quantum of the continuously varying space-time coordinates x_μ , and formulate Lagrange's equations in terms of a Lagrangian density \mathcal{L} such that Lagrangian L is given by

$$L = \int \mathcal{L} d^3x.$$

This is required relation.