VECTOR - ALGEBRA

SCALAR AND VECTOR PRODUCT OF FOUR VECTORS

5.) Scalar product of four vectors

show that $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

Proof :-

we have

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{a}\times\vec{b}\cdot\vec{m}$ where $\vec{m} = \vec{c}\times\vec{d}$

$= \vec{a}\cdot\vec{b}\times\vec{m}$, as dot and cross are interchangable in a scalar triple product.

$= \vec{a}\cdot\vec{b}\times(\vec{c}\times\vec{d})$

$= \vec{a}\cdot\{(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}\}$

$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

6.) Vector product of four vectors:-

show that

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$

(iii) $[\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$

(iv) If $\vec{a}, \vec{b}, \vec{c}$ are not coplanar and $\vec{r}$ any vector

then $\vec{r} = \dfrac{[\vec{r}\ \vec{b}\ \vec{c}]\,\vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\,\vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\,\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

Proof :- (i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$= \vec{n} \times (\vec{c} \times \vec{d})$

$= \vec{n} \cdot \vec{d}\,\vec{c} - \vec{n} \cdot \vec{c}\,\vec{d}$

$= [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}$ —— (1)

(ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$= (\vec{a} \times \vec{b}) \times \vec{m}$ where $\vec{m} = \vec{c} \times \vec{d}$

$= \vec{a}\ \vec{m}\ \vec{b} - \vec{b} \cdot \vec{m}\,\vec{a}$

$= \vec{a} \cdot \vec{c} \times \vec{d}\ \vec{b} - \vec{b} \cdot \vec{c} \times \vec{d}\ \vec{a}$

$= [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$ —— (2)

(iii) Equating the two values of $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ from (1) and (2) we get the required result (iii).

$[\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} = [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$

—— (3)

(iv) Putting $\vec{r}$ for $\vec{d}$ in (iii) we get

$$[\vec{a}\ \vec{b}\ \vec{c}]\vec{r} = [\vec{b}\ \vec{c}\ \vec{r}]\vec{a} - [\vec{a}\ \vec{c}\ \vec{r}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{r}]\vec{c}$$

$$= [\vec{r}\ \vec{b}\ \vec{c}]\vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\vec{c} \quad \text{—— (4)}$$

If $\vec{a}, \vec{b}, \vec{c}$ are not coplaner, then $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$

$\therefore$ Dividing (4) throughout by $[\vec{a}\ \vec{b}\ \vec{c}]$ we get

$$\vec{r} = \frac{[\vec{r}\ \vec{b}\ \vec{c}]\vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$