Runge-Kutta Methods →

As already mentioned, Euler's method is less efficient in practical problems since it requires h to be small for obtaining reasonable accuracy. The Runge-Kutta methods are designed are designed to give greater accuracy and they possess the advantage of requiring only the function values at some selected points on the subinterval.

If we substitute $y_1 = y_0 + hf(x_0, y_0)$ on the right side of Equation of Modified Euler's method, we obtain from $y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$ —— (I)

We obtain-

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0+h, y_0+hf_0)] \quad \text{—— (II)}$$

where $f_0 = f(x_0, y_0)$. If we now set

$$k_1 = hf_0 \quad \text{and} \quad k_2 = hf(x_0+h, y_0+k_1)$$

then the above equation becomes

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2) \quad \text{—— (III)}$$

which is the second-order Runge-Kutta formula. The error in this formula can be shown to be of order h^3 by expanding both sides by Taylor's series. Thus, the left side gives

$$y_0 + \frac{h}{\lfloor 1}y_0' + \frac{h^2}{\lfloor 2}y_0'' + \frac{h^3}{\lfloor 3}y_0''' \ldots$$

and on the right side

$$k_2 = hf(x_0+h, y_0+hf_0) = h\left[f_0 + h\frac{\partial f}{\partial x_0} + hf_0\frac{\partial f}{\partial y_0} + O(h^2)\right]$$

Since $\dfrac{df(x,y)}{dx} = \dfrac{\partial f}{\partial x} + f\dfrac{\partial f}{\partial y}$

we obtain

$$k_2 = h[f_0 + hf_0' + O(h^2)] = hf_0 + h^2f_0' + O(h^3)$$

So that the right side of Equation (III) gives

$$y_0 + \frac{1}{2}[hf_0 + hf_0 + h^2f_0' + O(h^3)] = y_0 + hf_0 + \frac{h^2}{2}f_0' + O(h^3)$$

$$= y_0 + hy_0' + \frac{h^2}{2}y_0'' + O(h^3)$$

It therefore follows that the Taylor series —

expansions of both sides of Eq (11) agree up to terms of order (h^3), which means that the error in this formula is of order h^3.

More, generally, If we set

$$\left.\begin{aligned} y_1 &= y_0 + W_1 k_1 + W_2 k_2 \\ k_1 &= h f_0 \\ k_2 &= h f(x_0 + \alpha_0 h, y_0 + \beta_0 k_1) \end{aligned}\right\} \quad \text{--- (4)}$$

then the Taylor series expansions of both sides of both sides of the both sides of the last equation (4) gives the identity.

$$y_0 + h f_0 + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f_0 \frac{\partial f}{\partial y}\right) + O h^3 = y_0 + (W_1 + W_2) h f_0 + W_2 h^2 \left(\alpha_0 \frac{\partial f}{\partial x} + \beta_0 f_0 \frac{\partial f}{\partial y}\right) + O(h^3)$$

Equating the coefficient of $f(x, y)$ and its derivatives on both sides, we obtain the relations

$$W_1 + W_2 = 1, \quad W_2 \alpha_0 = 1/2, \quad W_2 \beta_0 = 1/2 \quad \text{--- (5)}$$

Clearly $\alpha_0 = \beta_0$ and if α_0 is assigned any value arbitrarily, then the remaining parameters can be determined uniquely. If we set for example $\alpha_0 = \beta_0 = 1$, then we immediately obtain obtain $W_1 = W_2 = 1/2$, which gives formula (3)

It follows, therefore, that there are several second-order Runge-Kutta formulae and that formulae (4) and (5) constitute just one of several such formulae

Higher-order Runge-Kutta formulae exist, of which we mention only the fourth-order formula defined by.

$$y_1 = y_0 + W_1 k_1 + W_2 k_2 + W_3 k_3 + W_4 k_4$$

where $k_1 = h f(x_0, y_0)$

$k_2 = h f(x_0 + \alpha h, y_0 + \beta_0 k_1)$

$$k_3 = hf(x_0+\alpha_1 h,\ y_0+\beta_1 k_1+v_1 k_2)$$
$$k_4 = hf(x_0+\alpha_2 h,\ y_0+\beta_2 k_1+v_2 k_2+\delta_1 k_3) \quad \text{--- (6)}$$

Where the parameters have to be determined by expanding both sides of the first equation (6) by Taylor's series and securing agreement of terms up to and including those containing h^4. The choice of the parameters is again, arbitrary and we have therefore several fourth-order Runge-Kutta formulae. If for example, we set

(7)
$$\begin{cases} \alpha_0=\beta_0=\frac{1}{2} & \alpha_1=\frac{1}{2} & \alpha_2=1 \\ \beta_1=\frac{1}{2}(\sqrt{2}-1) & \beta_2=0 & \\ v_1=1-\frac{1}{\sqrt{2}} & v_2=-\frac{1}{\sqrt{2}} & \delta_1=1+\frac{1}{\sqrt{2}} \\ W_1=W_4=\frac{1}{6} & W_2=\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right) & W_3=\frac{1}{3}\left(1+\frac{1}{\sqrt{2}}\right) \end{cases}$$

We obtain the method of Gill, whereas the choice

$$\left.\begin{array}{ll} \alpha_0=\alpha_1=\frac{1}{2}, & \beta_0=v_1=\frac{1}{2} \\ \beta_1=\beta_2=v_2=0 & \alpha_2=\delta_1=1 \\ W_1=W_4=\frac{1}{6} & W_2=W_3=\frac{2}{6} \end{array}\right\} \text{--- (8)}$$

leads to the fourth-order Runge-Kutta formula, the most commonly used one in practice.

$$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4) \quad \text{--- (9)}$$

Where
$$\left.\begin{array}{l} k_1=hf(x_0,y_0) \\ k_2=hf(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_1) \\ k_3=hf(x_0+\frac{1}{2}h,\ y_0+\frac{1}{2}k_2) \\ k_4=hf(x_0+h,\ y_0+k_3) \end{array}\right\} \text{--- (10)}$$

In which the error is of order h^5. Complete derivation of the formula is exceeding complicated, and the interested reader is referred of book by Levy & Baggot.

* <u>Working Rule</u> → for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from

$$\frac{dy}{dx} = f(x, y), \quad y(0) = y_0 \text{ at } x_0$$

is as follows

Calculate successively.

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1)$$
$$k_3 = hf(x_0 + \tfrac{1}{2}h, y + \tfrac{1}{2}k_2)$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$

and finally compute

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

which gives the required approximate value of

$$y_1 = y_0 + k.$$