*. Rule for finding the Particular Integral (P.I)

(c) Case-III When $X = x^m$

then $P.I = \frac{1}{f(D)} x^m = [f(D)]^{-1} \cdot x^m$

* Expand $[f(D)]^{-1}$ in ascending power of D as far as the term in D^m and operate x^m by term. Since the $(m+1)^{th}$ and higher derivatives of x^m are zero.

Formulae to expand $[f(D)]^{-1}$

(i) $(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 + \cdots$

(ii) $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

(iii) $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots (1+m) D^m$

(iv) $(1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots \Sigma m D^m$

(v) $(1+D)^m = 1 + \frac{m}{\lfloor 1} D + \frac{m(m-1)}{\lfloor 2} D^2 + \cdots \frac{[m(m-1)(m-2)\cdots 1]}{\lfloor m} \cdot D^m$

(d) Case (IV) :- When $X = e^{ax} V$, where V is a function of x.

$$\frac{1}{f(D)} e^{ax} V = e^{ax} \cdot \frac{1}{f(D+a)} V$$

(e) Case (V) : When X is any other function of x.

$$P.I = \frac{1}{f(D)} X$$

if $f(D) = (D-\lambda_1)(D-\lambda_2)(D-\lambda_3) \cdots (D-\lambda_n)$ into partial fractions. then $\frac{1}{f(D)} = \frac{a_1}{(D-\lambda_1)} + \frac{a_2}{(D-\lambda_2)} + \cdots \frac{a_n}{(D-\lambda_m)}$

$$P.I = \frac{1}{f(D)} X = \left[\frac{a_1}{(D-\lambda_1)} + \frac{a_2}{D-\lambda_2} + \cdots \frac{a_m}{D-\lambda_m}\right] X.$$

Then $P.I = \frac{1}{f(D)} X = a_1 e^{\lambda_1 x} \int X \cdot e^{-\lambda_1 x} dx + a_2 e^{\lambda_2 x} \int X \cdot e^{-\lambda_2 x} dx$

$+ \dots + a_m e^{\lambda_m x} \int X \cdot e^{-\lambda_m x} dx$

$\left[\text{from } \because \frac{1}{D-a} X = e^{ax} \int X \cdot e^{-ax} dx \right]$

Exp: Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 0 \cdot y = x^2 + 2x + 4$

Sol: Given differential Equation is non-homo-geneous. Then general solution is

$y = C.F + P.I$

To find C.F from

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Then auxiliary Eqn. $D^2 + D = 0 \Rightarrow D(D+1) = 0$

$\Rightarrow D = 0 \;\&\; D = -1$

$C.F. = c_1 \cdot e^{0 \cdot x} + c_2 \cdot e^{-x} = c_1 + c_2 e^{-x}$

and $P.I = \frac{1}{D(D+1)} \cdot (x^2 + 2x + 4)$

$= \frac{1}{D} \cdot (D+1)^{-1} [x^2 + 2x + 4]$

$= \frac{1}{D} [1 - D + D^2 - D^3 + D^4 - \dots] (x^2 + 2x + 4)$

$= \left(\frac{1}{D} - 1 + D - D^2 + D^3 - \dots\right) (x^2 + 2x + 4)$

$= \frac{1}{D}(x^2 + 2x + 4) - (x^2 + 2x + 4) + D(x^2 + 2x + 4)$

$- D^2(x^2 + 2x + 4) + D^3(x^2 + 2x + 4) - \dots$

$$\Rightarrow \int (x^2+2x+4)\,dx - (x^2+2x+4) + \frac{d}{dx}(x^2+2x+4)$$

$$- \frac{d^2}{dx^2}(x^2+2x+4) + \frac{d^3}{dx^2}(x^2+2x+4)$$

$$= \frac{x^3}{3} + 2\cdot\frac{x^2}{2} + 4x - (x^2+2x+4) + (2x+2\cancel{x}1+0)$$

$$- \frac{d}{dx}(2x+2\cdot 1+0) + \frac{d^2}{dx}(2x+2\cdot 1+0)$$

$$= \frac{x^3}{3} + \cancel{x^2} + 4x - \cancel{x^2} - \cancel{2x} - 4 + \cancel{2x} + \cancel{2} - (\cancel{2}\cdot 1+0)+$$

$$P.I = \frac{x^3}{3} + 4x - 4$$

Thus.

$$G.S = C.F + P.I$$

$$= c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x - 4$$