ABSTRACT ALGEBRA

ROOTS OF A POLYNOMIAL IN A EXTENSION FIELD

Definition 1 Let K be an extension of a field F and let $F[x]$ be the ring of polynomials in x over F. An element $\alpha \in K$ is said to be a root of a polynomial $f(x) \in F[x]$ in K if $f(\alpha) = 0$.

Definition 2 Let K be an extension of a field F. Then an element $\alpha \in K$ is said to be a root of multiplicity m of the polynomial $f(x) \in F[x]$ if in $K[x]$, $f(x) = (x-\alpha)^m g(x)$ provided $g(\alpha) \neq 0$, where $g(x) \in F[x]$.

If $m = 1$, then α is called a simple root.

Definition 3 Let F be a field and let $F[x]$ be the ring of polynomials in x over F. Since we know that $F[x]$ is an integral domain with unity containing F as a proper subring.

A polynomial $f(x) \in F[x]$ is called irreducible over F if whenever $f(x) = p(x)\, q(x)$ with $p(x), q(x) \in F[x]$ then either $p(x)$ or $q(x)$ has degree zero i.e. either $p(x) \in F$ or $q(x) \in F$

If $f(x)$ is not irreducible, then it is called reducible.

Theorem Remainder theorem Let K be an extension of a field F. Then for each polynomial $f(x) \in F(x)$ and for any element $\alpha \in K$ there exists a polynomial $q(x) \in K[x]$ such that

$$f(x) = (x-\alpha)\, q(x) + f(\alpha) \text{ where deg. } q(x) = \text{deg. } f(x) - 1$$

Proof:- Since $F \subseteq K$, then $F[x] \subseteq K[x]$

i.e. every polynomial over F can also be considered to be the polynomial over K.

Also $\alpha \in K$, then $(x-\alpha) \in K[x]$ is a polynomial of degree 1.

If $f(x) \in F[x]$, then by division alogrithm there exists two polynomial $q(x)$ and $r(x)$ in $K[x]$ such that

$$f(x) = (x-\alpha)\, q(x) + r(x) \quad \text{--- (1)}$$

where either $r(x) = 0$ or $\deg r(x) < \deg(x-\alpha) = 1$

i.e. either $r(x) = 0$ or $\deg. r(x) = 0$

$$[\because \deg r(x) < 1]$$

Thus $r(x)$ is a constant polynomial, let $r(x) = c$.

Now from (1), we have

$$f(x) = (x-\alpha)\, q(x) + c \quad \text{--- (2)}$$

$$\Rightarrow f(\alpha) = c$$

$$\therefore f(x) = (x-\alpha)\, q(x) + f(\alpha)$$

Also $\deg. f(x) = \deg. [(x-\alpha)\, q(x)]$

$$\Rightarrow \deg q(x) = \deg. f(x) - 1$$

Hence the theorem.