Rolle's Theorem

Q. State and prove Rolle's Theorem

Statement:- If a function $f(x)$ be defined such that

(i) $f(x)$ is continuous at every point of the closed interval $[a, b]$

(ii) $f'(x)$ exists at every point of the open interval $]a, b[$

(iii) $f(a) = f(b)$,

then there exists at least one value of x, say c such that $f'(x) = 0$ where $a < c < b$

Proof:

Since $f(x)$ is continuous in $[a, b]$, therefore $f(x)$ is bounded in $[a, b]$ & moreover it attains its bounds at least once in $[a, b]$

Let M & m be the l.u.b. and g.l.b. of $f(x)$ in $[a, b]$ respectively.

If M and m are equal, then we have

$M = m = f(a) = f(b) = f(x) \ \forall \ x \in [a, b]$

Thus $f(x)$ is constant $\forall \ x \in [a, b]$

& hence $f'(x) = 0 \ \forall \ x \in [a, b]$

Consequently we have $f'(c) = 0$ for any point $c \in]a, b[$

Again if $M \neq m$, then at least one of M & m must be different from $f(a) = f(b)$

Let $M \neq f(a) = f(b)$

Since $f(x)$ is continuous in $[a,b]$ & M be its l.u.b.,
therefore there is at least one point c in $]a,b[$ such that
$f(c) = M$.

Since M is l.u.b. of $f(x)$, therefore $f(x) \leq M \; \forall x \in [a,b]$

i.e. $f(x) \leq f(c)$, where $c \in]a,b[$

i.e. $f(c \pm h) \leq f(c)$ where $c \pm h \in]a,b[$ & h is positive

i.e. $f(c \pm h) - f(c) \leq 0$

$$f(c+h) - f(c) \leq 0 \Rightarrow \frac{f(c+h)-f(c)}{h} \leq 0 \quad \text{--- (1)}$$

$$\& \; f(c-h) - f(c) \leq 0 \Rightarrow \frac{f(c-h)-f(c)}{-h} \geq 0 \quad \text{--- (2)}$$

Taking limit of (1) & (2) as $h \to 0$, we get

$$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} \leq 0 \quad \text{i.e. } Rf'(c) \leq 0 \quad \text{--- (3)}$$

$$\& \; \lim_{h \to 0} \frac{f(c-h)-f(c)}{-h} \geq 0 \quad \text{i.e. } Lf'(c) \geq 0 \quad \text{--- (4)}$$

Since from condition (ii) of the theorem $f'(x)$ exists at every point of $]a,b[$, then we should have $Rf'(c) = Lf'(c)$ for $c \in]a,b[$ & hence from (3) & (4) we have

$$Rf'(c) = 0 = Lf'(c)$$

i.e. $f'(c) = 0$ where $a < c < b$

Similarly taking $f(c) = m$ we can prove $f'(c)$ as above.