Abstract Algebra

RINGS :-

Skew field :- A ring R with unit element having at least two elements is called a skew field if every non-zero elements of R possess their multiplicative inverse. The skew field is also known as division ring.

Notes:-

(i) We can define the field as a commutative division ring.

(ii) Every field is also a division ring.

(iii) A skew-field has no zero divisors.

Theorem :— Every finite integral domain is a field.

Proof :- Let D be a finite integral domain. Therefore by the definition of integral domain we have that D is a commutative ring with unit element having no zero divisors. Let $D = \{a_1, a_2, \ldots a_n\}$. In order to show that D is a field we only have to show that D has multiplicative inverse for every non-zero element in D. For this purpose let $a \neq 0$ be any arbitrary element of D and consider the set.

$$D_1 = \{aa_1, aa_2, \ldots, aa_n\}$$

D_1 has n distinct products. For this let us suppose $aa_i = aa_j$ for $i \neq j$

$$\Rightarrow a(a_i - a_j) = 0 \quad \text{(By left distributive law)}$$

Since D has no zero divisors and $a \neq 0$
then

$$a_i - a_j = 0 \quad \text{for } i \neq j$$

$$\Rightarrow \quad a_i = a_j \quad \text{for } i \neq j$$

$\Rightarrow$ This is a contradiction because D has n distinct elements $a_1, a_2, \ldots, a_n$.
Consequently D_1 has n distinct products.
But the elements of D_1 are the elements of D placed in some order. Further D has unit element, that is

$$1 \in D$$

$$\Rightarrow \quad 1 \in D_1$$

This implies that there exists an element b in D such that

$$ab = 1 \, .$$

But D is commutative. Therefore

$$ab = 1 = ba$$

Thus a^{-1} exists in D. Hence D is a field.