Remaining part of Kroncker's theorem

ϕ preserves the Compositions. Let $a, b \in F$, then

$$\phi(a+b) = (f(x)) + (a+b)$$
$$= (f(x)) + a) + ((f(x)) + b) = \phi(a) + \phi(b)$$

and
$$\phi(ab) = f(x) + ab = ((f(x)) + a)((f(x)) + b)$$
$$= \phi(a)\phi(b)$$

Hence K' is a subfield of K- isomorphic to F. From above isomorphism, we can say that every element of K' is corresponding element of F, so that F is regarded as subfield of K, thus K is an extension of a field F.

Now we show that $f(x)$ has a root in K.

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ with $a_n \neq 0$ and $a_0, a_1, a_2, \dots, a_n \in F$

Since $f(x) \in (f(x))$

$$\Rightarrow (f(x)) + f(x) = (f(x))$$
$$\Rightarrow ((f(x)) + (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n) = (f(x))$$
$$\Rightarrow ((f(x)) + a_0) + ((f(x)) + a_1 x) + \dots + ((f(x)) + a_n x^n) = (f(x))$$
$$\Rightarrow a_0((f(x)) + 1) + a_1((f(x)) + x) + \dots + a_n((f(x)) + x)^n = ((f(x))$$

But $(f(x))$ is the zero element of $F[x]/(f(x)) = K$

Thus each element $(f(x)) + x$ in K satisfies the polynomial $f(x)$, this show that $f(x)$ has a root in K.

Finally we show that $[K:F] = \deg. f(x)$

For which we shall show that the set

$$S = \{(f(x)) + 1), ((f(x)) + x), \dots, ((f(x)) + x^{n-1})\}$$

forms a basis of K over F.

S is linearly independent.
For $a_0, a_1, a_2, \ldots, a_{n-1} \in F$, we have
$$a_0((f(x))+1) + a_1((f(x))+x) + \cdots + a_{n-1}((f(x))+x^{n-1}) = (f(x))$$
$$\Rightarrow ((f(x))+a_0) + ((f(x))+a_1x) + \cdots + ((f(x))+a_{n-1}x^{n-1}) = f(x)$$
$$\Rightarrow (f(x)) + (a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) = (f(x))$$
$$\Rightarrow a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \in (f(x))$$
$$\Rightarrow a_0 + a_1x + \cdots + a_{n-1}x^{n-1} = f(x)g(x) \text{ for some } g(x) \in K[x].$$
Since dg. r $f(x) = n$, but L.H.S. is a polynomial of degree $n-1$, so that
$$g(x) = 0$$
$$\Rightarrow a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} = 0$$
$$\Rightarrow a_0 = 0 = a_1 = \cdots = a_{n-1}$$
Thus S is linearly independent.
Next, we show that S generates $K(F)$. For which let $(f(x)) + P(x)$ be any element of $K = \dfrac{F[x]}{(f(x))}$, where $p(x) \in F(x)$ is an arbitrary element.
By the division alogrithm, there exists two polynomial $q(x)$ and $r(x)$ in $F[x]$ such that
$$p(x) = f(x)q(x) + r(x)$$
where either $r(x) = 0$ or dg. $r(x) <$ dg. $f(x)$
so that
$$(f(x)) + p(x) = (f(x)) + [f(x)q(x) + r(x)]$$
$$= [(f(x)) + f(x)q(x)] + [(f(x)) + r(x)]$$
$$= [(f(x)) + 0] + [(f(x)) + r(x)]$$
$$[\because f(x)q(x) \in (f(x))]$$

$= (f(x)) + r(x)$

Now, deg $r(x) <$ deg. $f(x)$, so assume $r(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}$ for $b_0, b_1, \ldots, b_{n-1} \in F$.

$\therefore (f(x)) + p(x) = (f(x)) + b_0 + b_1 x + \cdots + b_{n-1}x^{n-1})$

$= ((f(x)) + b_0) + (f(x)) + b_1 x)$

$+ \cdots + ((f(x)) + b_{n-1}x^{n-1})$

$= b_0((f(x)) + 1) + b_1(f(x)) + x) + \cdots + b_{n-1}((f(x)) + x^{n-1})$

Thus $(f(x)) + p(x)$ is a linearly Combination of the elements of S, since $p(x)$ is an arbitrary element of $F[x]$.

Hence S generates K.

Consequently S forms a basis of K over F.

$\therefore$ dim $K(F)$ = number of elements in $S = n$

or, $[K : F] = n =$ deg. $f(x)$

Hence the theorem.