Normal series of a group :-

A finite sequence of subgroups $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_k = (e)$ of a group G is called a subnormal series of G if G_{i-1} is a normal subgroup of G_i, $\forall\, i = 0, 1, 2, \ldots, k-1$.

The quotient groups G_i/G_{i+1} are called the factor groups of the subnormal series.

If each G_i is a normal subgroup of G itself, then the series is said to be a normal series of G.

Q.) Prove that every abelian group is solvable.

Sol:- Let G is an abelian group. Let $G = N_0$ and $N_1 = (e)$

Then $G = N_0 \supseteq N_1 = (e)$ is a solvable series of G.

Clearly $N_1 = (e)$ is a normal subgroup of $N_0 = G$. Since for any $a \in G$ we have

$$aea^{-1} = a^{-1}a = e \in (e) = N_1$$

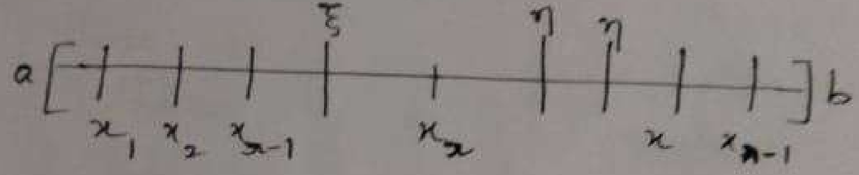
Since G is abelian, so the quotient group $G/N_1 = G(e) = G$ is abelian.

As every quotient group of an abelian group is abelian. So G is solvable.

Proved

Let now ξ, η be any two points in $[a, b]$ such that $|\xi - \eta| \neq \delta$.
If these two points are in the same sub-interval, then by 1, we have

$$|f(\xi) - f(\eta)| < \frac{1}{2}\epsilon$$



If ξ, η do not belong to the same sub-interval, then surely they lie one in each of the two consecutive intervals. If x_r is the point of division such that $x_{r-1} < \xi < x_r < \eta < x_{r+1}$, then we have

$$|f(\xi) - f(\eta)| = |f(\xi) - f(x_r) + f(x_r) - f(\eta)|$$
$$\leq |f(\xi) - f(x_r)| + |f(x_r) - f(\eta)|$$
$$< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \quad \text{by } 1$$

Thus we have shown that, given $\epsilon > 0$ there exists $\delta > 0$

$$|f(\xi - f(\eta)| < \epsilon$$

for any two points ξ, η in $[a, b]$ such that $|\xi - \eta| < \delta$.
Hence f is uniformly continuous in $[a, b]$.

Example :-

① show that the function $f(x)=\frac{1}{x} (x>0)$. is uniformly Continuous in $]0,1]$.

sol:- We suppose if possible, that f is uniformly Continuous in $]0,1]$. Then, given $\epsilon>0$, there can be found $\delta>0$ independent of any choice x_0 in $]0,1]$ such that

$$\left|\frac{1}{x}-\frac{1}{x_0}\right|<\epsilon \text{ for } |x-x_0|<\delta$$

If we choose $x_0=\delta$ itself. we see that the interval

$$]x_0-\delta, x_0+\delta[$$

becomes $]0,2\delta[$ and the condition ① now must hold for any interval. But then

$$\frac{1}{x}-\frac{1}{\delta}=\frac{\delta-x}{x\delta} \longrightarrow \infty \text{ as } x \to 0$$

and so by choosing x sufficiently near 0, the condition ① will violated. Thus f is non-uniformly Continuous in $]0,1]$. If however, uniformly Continuous in $[c,1]$ where $c>0$, since Continuous in this closed interval.