Srinivasa Ramanujan (1887-1920) described an iterative procedure to determine the smallest root of the equation $f(x)=0$ —— (1)

where $f(x)$ is of the form

$$f(x) = 1-(a_1x+a_2x^2+a_3x^3+\cdots\cdots) \quad \text{—— (2)}$$

To explain the method of procedure, we consider the quadratic equation

$$f(x) = a_0x^2+a_1x+a_2=0$$

with the roots x_1 and x_2, such that $|x_1|<|x_2|$.

Then the equation defined by

$$\phi(x) = a_2x^2+a_1x+a_0=0$$

$$\Rightarrow \quad 1+\frac{a_1}{a_0}x+\frac{a_2}{a_0}x^2=0$$

will have roots $\frac{1}{x_1}$ and $\frac{1}{x_2}$ such that

$$\frac{1}{|x_1|}>\frac{1}{|x_2|}$$

Now

$$\frac{1}{\phi(x)} = \left(1+\frac{a_1}{a_0}x+\frac{a_2}{a_0}x^2\right)^{-1}$$

$$\left(1+\frac{a_1}{a_0}x+\frac{a_2}{a_0}x^2\right)^{-1} = \frac{k_1}{x-\frac{1}{x_1}}+\frac{k_2}{x-\frac{1}{x_2}}$$

$$= \frac{-k_1x_1}{1-xx_1}+\frac{-k_2x_2}{1-xx_2}$$

$$= -k_1x_1(1-xx_1)^{-1}-k_2x_2(1-xx_2)^{-1}$$

$$= \sum_{i=0}^{\infty} b_ix^i \quad \text{where}$$

$$b_i = -\sum_{r=1}^{2} k_rx_r^{i+1}$$

Then

$$\frac{b_{i-1}}{b_i} = \frac{k_1 x_1^i + k_2 x_2^i}{k_1 x_1^{i+1} + k_2 x_2^{i+1}} = \frac{\frac{k_1}{k_2}\left(\frac{x_1}{x_2}\right)^i + 1}{\frac{k_1}{k_2}\left(\frac{x_1}{x_2}\right)^{i+1} + 1} \cdot \frac{1}{x_2}$$

Since $\frac{x_1}{x_2} < 1$, it follows that

$$\lim_{i \to \infty} \frac{b_{i-1}}{b_i} = \frac{1}{x_2}$$

which is the smallest root. This is the basis of Ramanujan's method which is outlined below:

To find the smallest root of $f(x)=0$, we consider $f(x)$ in the form

$$f(x) = 1 - (a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

and then write

$$[1 - (a_1 x + a_2 x^2 + a_3 x^3 + \cdots)]^{-1} = b_1 + b_2 x + b_3 x^2 + \cdots \quad (3)$$

$$\Rightarrow 1 + (a_1 x + a_2 x^2 + a_3 x^3 + \cdots) + (a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2 + \cdots \quad (4)$$
$$= b_1 + b_2 x + b_3 x^2 + \cdots$$

To find b_i, we equate coefficients of like powers of x on both sides of Eq(4). We then obtain

$$\left.\begin{aligned} b_1 &= 1 \\ b_2 &= a_1 = a_1 b_1 \quad \text{since } b_1 = 1 \\ b_3 &= a_2 + a_1^2 = a_2 b_1 + a_1 b_2, \quad \text{since } b_2 = a_1 \\ &\vdots \\ b_k &= a_1 b_{k-1} + a_2 b_k + \cdots + a_{k-1} b_1 \\ b_k &= a_{k-1} b_1 + a_{k-2} b_2 + \cdots a_1 b_{k-1} \end{aligned}\right\} \quad (5)$$

The ratios $\frac{b_{i-1}}{b_i}$, called the convergents, approach, in the limit, the smallest root of $f(x)=0$.

Exp. Find the smallest root of the equation

$$f(x) = x^3 - 9x^2 + 26x - 24 = 0$$

We have

$$f(x) = 1 - \frac{26}{24}x + \frac{9}{24}x^2 - \frac{1}{24}x^3$$

$$f(x) = 1 - \left(\frac{13}{12}x - \frac{3}{8}x^2 + \frac{1}{24}x^3\right)$$

Here $a_1 = \frac{13}{12}$, $a_2 = -\frac{3}{8}$, $a_3 = \frac{1}{24}$, $a_4 = a_5 = 0 \ldots$

Now $b_1 = 1$

$$b_2 = a_1 = \frac{13}{12} = 1.0833$$

Therefore, $\Rightarrow \frac{b_1}{b_2} = \frac{12}{13} = 0.923$

$$b_3 = a_1 b_2 + a_2 b_1 = \frac{13}{12}\left(\frac{13}{12}\right) + \left(\frac{-3}{8}\right)(1)$$

$$b_3 = 0.7986$$

$$\Rightarrow \because \frac{b_2}{b_3} = \frac{1.0833}{0.7986} = 1.356$$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = \frac{13}{12}(0.7986) + \left(\frac{-3}{8}\right)(1.0833) + \frac{1}{24}(1)$$

$$= (1.0833)(0.7986) + \left(\frac{-3}{8}\right)(1.0833) + \frac{1}{24}(1)$$

$$b_4 = 0.5007$$

$$\Rightarrow \because \frac{b_3}{b_4} = \frac{0.7986}{0.5007} = 1.595$$

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 \quad (\because a_4 = 0 \Rightarrow a_4 b_1 = 0$$

$$= (1.0833)(0.5007) + \left(\frac{-3}{8}\right)(0.7986) + \frac{1}{24}(1.0833)$$

$$b_5 = 0.2880$$

Therefor $\frac{b_4}{b_5} = \frac{0.5007}{0.2880} = 1.7382$

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + \underbrace{a_4 b_2 + a_5 b_1}_{0} \quad \because a_4 = a_5 = 0$$

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3$$

$$b_6 = 0.1575$$

Therefore

$$\frac{b_5}{b_6} = \frac{0.2880}{0.1575} = 1.8286$$

$$b_7 = a_1 b_6 + a_2 b_5 + a_3 b_4 + \underbrace{a_4 b_3 + a_5 b_2 + a_6 b_1}_{0} \quad \because a_4 = a_5 \ldots$$

$$b_7 = 0.0835$$

Therefore $\frac{b_6}{b_7} = \frac{0.1575}{0.0835} = 1.8862$

$$b_8 = a_1 b_7 + a_2 b_6 + a_3 b_5 + \underbrace{a_4 b_4 + \cdots}_{0}$$

$$b_8 = 0.0434$$

Therefore $\frac{b_7}{b_8} = \frac{0.0835}{0.0434} = 1.9240$

$$b_9 = a_1 b_8 + a_2 b_7 + a_3 b_6 + \underbrace{a_4 b_5 + \cdots}_{0}$$

$$b_9 = 0.0223$$

Therefore $\frac{b_8}{b_9} = \frac{0.0434}{0.0223} = 1.9462$

$$b_{10} = a_1 b_9 + a_2 b_8 + a_3 b_7 + \underbrace{a_4 b_6 + \cdots}_{} \quad \because a_4 = a_5 = a_6 = \cdots = 0$$

$$b_{10} = (1.0833)(0.0223) + \left(\frac{-3}{8}\right)(0.0434) + \frac{1}{24}(0.0835) + 0$$

$$b_{10} = 0.0129$$

The root of the given epuation are 2,3,4 and it can be seen that the successive convergents approach the value 2.