Abstract Algebra

RADICALS WITH PRIME EXPONENTS :-

In order to obtain the radical extensions we may need only radicals with prime exponents. We Consider an equation

$$x^n - a = 0 \quad \text{—— (1)}$$

If $n = rs$, then we have

$$x^{rs} - a = 0 \quad \text{or} \quad (x^r)^s - a = 0 \quad \text{—— (2)}$$

Now a root α of $x^n - a = 0$ is a root of

$$x^r = \beta \quad \text{—— (3)}$$

where β is a root of $x^s - a = 0$ —— (4)

Thus, if F is a field such that $x^n - a \in F[x]$, then

$$F \subseteq F(\beta) \subseteq F(\alpha)$$

Proceeding in this manner, we can find a finite series of pure extensions such that each member of the series is obtained on adjoining to its predecessor a radicals with a prime exponent.

Now we shall refer to solvability by radicals in above manner as follows:

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the n distinct roots of an equation $x^n - a = 0$ over F, which are in its splitting field.

Obviously, none of α_i is zero. Then we Consider $1 = \frac{\alpha_1}{\alpha_1}, \frac{\alpha_1}{\alpha_2}, \ldots, \frac{\alpha_1}{\alpha_n}$ as the n distinct roots of an equation $x^n - 1 = 0$.

These roots are also in the splitting field of $x^n - a = 0$.

Hence, we conclude that the splitting field of $x^n - 1 = 0$ may be regarded as as intermediate field between F and the splitting field of $x^n - a = 0$.

ROOTS OF UNITY AND CYCLOTOMIC POLYNOMIALS

Definition 1 Let F be a field. Then the roots of $x^n - 1 = 0$ over F are known as the primitive n^{th} roots of unity in its splitting field if $\omega^n = 1$, but $\omega^m \neq 1$ for any positive integer $m < n$. There are exactly $\phi(n)$ primitive n^{th} roots of unity for every integer n.

Definition 2 If we consider the equation $x^n - 1 = 0$ over the field of rational numbers, then the complex numbers satisfying $x^n - 1 = 0$ divide the circumference of a circle with centre at the origin and of unit radius into n equal parts. Thus the equation $x^n - 1 = 0$ is known as cyclotomic.