* Further tests of convergence by using Ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = k$$

But it is often convenient to state the ratio test as follows:-

In the positive term series $\sum a_n$,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = k$$

then the series convergence $k>1$ and divergence to $k<1$ but the test fails for

(11) Part of Exp. Ignoring first term of ser. we have

$$a_n = \frac{x^n}{n^2+1} \quad \& \quad a_{n+1} \frac{x^{n+1}}{(n+1)^2+1}$$

then By Ratiotest

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1}{x} \cdot \frac{(n+1)^2+1}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{x} \cdot \frac{n^2\{(1+\frac{1}{n})^2+\frac{1}{n^2}\}}{n^2(1+1/n^2)}$$

$$= \frac{1}{x}$$

$\because \lim_{n\to\infty} \frac{1}{n^2}$

$\because \lim_{n\to\infty} \frac{1}{n} = 0$

By Ratiotest $\sum a_n$ converges if $\frac{1}{x}>1$ $\Rightarrow$ $x<1$ and $\sum a_n$ diverges if $x>1$. Ratio test fails if $x=1$

for $x=1$, $a_n = \frac{1}{n^2+1}$, since $\frac{1}{n^2+1} < \frac{1}{n^2}$ $\forall n \in N$

~~Since $\sum \frac{1}{n^2}$ converges $\Rightarrow$ Σ 1 comparision test $\Sigma \frac{1}{n^2}$~~ converse. Hence $\sum a_n$ converges for $x \leq 1$ & div. for

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* When the Ratio Test fails, we apply the following test

Remark

(1) Raabe's Test:- In the positive term series $\sum a_n$ if

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = k$$

then the series convergence $k>1$ and divergence for $k<1$, but test fails for $x=1$.

Proof:- (i) Case I:- When $k>1$, let a number p such that $k>p>1$ and compare $\sum a_n$ with the series $\sum \frac{1}{n^p}$ which is convergent since $p>1$.

$$a_n = \frac{1}{n^p} \quad \& \quad a_{n+1} = \frac{1}{(n+1)^p}$$

$\therefore$ $\sum a_n$ will converge, If form and after some term

$$\frac{a_n}{a_{n+1}} > \frac{(n+1)^p}{n^p} > 1$$

or

$$\frac{a_n}{a_{n+1}} > \left(1+\frac{1}{n}\right)^p$$

or

$$\frac{a_n}{a_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2 \cdot n^2} + \cdots\cdots$$

or

$$\frac{a_n}{a_{n+1}} - 1 > \frac{p}{n} + \frac{p(p-1)}{n^2} + \cdots\cdots$$

or if $n\left(\frac{a_n}{a_{n+1}}-1\right) > p + \frac{p(p-1)}{2n} + \cdots$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) > \lim_{n\to\infty}\left(p+\frac{p(p-1)}{2n}+\cdots\right)$$

$k > p$ which is true

Hence Σa_n is convergent

(ii) Case II when $k<1$ can be proved similarly.

(2) Logarithmic test:-

In the positive term series Σa_n if

$$\lim_{n\to\infty}\left[n \log \frac{a_n}{a_{n+1}}\right] = k$$

then the series converges for $k>1$ and diverges for $k<1$, but the test fails for $k=1$.

Proof:- Its proof is similar to that of Raabe's Test.

$\because$ Σa_n will converge if form

$$\frac{a_n}{a_{n+1}} > \frac{(n+1)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p$$

or $\log\left(\frac{a_n}{a_{n+1}}\right) > \log\left(1+\frac{1}{n}\right)^p$

or $\log\left(\frac{a_n}{a_{n+1}}\right) > p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \right.$

$\therefore \quad n\log\left(\frac{a_n}{a_{n+1}}\right) > \left(p - \frac{p}{2n} + \frac{p}{3n^2} - \cdots\right)$

$k > p$ which is true

Hence $\sum a_n$ is convergent.

* Consider the two series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$, where first series is divergent and second series is convergent, but for both the series

$$\lim_{n\to\infty}\left[n\left(\frac{a_n}{a_{n+1}} - 1\right)\right] = 1$$

<u>Remark</u>. Raabe's Test is stronger than Ratio test.

<u>Exp</u>. Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{1.3.5 \cdots\cdots (2n-1)}{2.4.6 \cdots\cdots 2n} \cdot \frac{1}{n}$$

<u>Solution</u> We have

$$a_n = \frac{1.3.5 \cdots\cdots (2n-1)}{2.4.6 \cdots\cdots 2n} \cdot \frac{1}{n}$$

$$a_{n+1} = \frac{1.3.5 \cdots\cdots (2n-1)(2n+1)}{2.4.6 \cdots\cdots 2n\,(2n+2)} \cdot \frac{1}{(n+1)}$$

Then $\frac{a_n}{a_{n+1}} = \frac{(2n+2)(n+1)}{(2n+1)\cdot n}$ — (1)

$$= \frac{2n(1+\frac{1}{n})\cdot n(1+\frac{1}{n})}{2n(1+\frac{1}{2n})\cdot n}$$

$$= (1+\frac{1}{n})\frac{(1+1/n)}{(1+1/2n)}$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{(1+\frac{1}{n})(1+1/n)}{(1+1/2n)}$$

$\because \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = 1$ means that

Ratio test fails.

So, we apply Raabe's test

we have. $\frac{a_n}{a_{n+1}} = \frac{(2n+2)(n+1)}{(2n+1)\cdot n}$

from (1)

$$n\left[\frac{a_n}{a_{n+1}}-1\right] = n\left[\frac{(2n+2)(n+1)}{n(2n+1)}-1\right]$$

$$= \frac{3n+2}{2n+1} = \frac{(3+2/n)}{(2+1/n)}$$

$$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}}-1\right] = \lim_{n\to\infty}\frac{(3+2/n)}{(2+1/n)}$$

$\lim_{n\to\infty} n\left[\frac{a_n}{a_{n+1}}-1\right] = \frac{3}{2} > 1$, Hence,

given series converges.