Q. Prove that any two isomorphic integral domains have isomorphic quotient fields. ③

Let D & D' be two isomorphic integral domains. Let f be an isomorphism of D onto D'.

Therefore if a, b, c, d are the elements of D, then $f(a), f(b), f(c), f(d)$ will be the elements of D'.

Also $f(a+b) = f(a) + f(b)$
& $f(ab) = f(a)f(b)$ for all $a, b \in D$

Let F & F' be the quotient fields of D & D' respectively.

Then $F = \{\frac{a}{b} : a, b \in D \text{ \& } b \neq 0\}$
& $F' = \{\frac{f(a)}{f(b)} : f(a), f(b) \in D \text{ \& } f(b) \neq 0\}$

Our purpose is to prove that $F \cong F'$

Let us consider a mapping $\phi : F \to F'$ such that
$\phi\left(\frac{a}{b}\right) = \frac{f(a)}{f(b)} \quad \forall \frac{a}{b} \in F$

ϕ is one-one because $\phi\left(\frac{a}{b}\right) = \phi\left(\frac{c}{d}\right) \Rightarrow \frac{f(a)}{f(b)} = \frac{f(c)}{f(d)}$
$\Rightarrow f(a)f(d) = f(b)f(c)$
$\Rightarrow f(ad) = f(bc)$
$\Rightarrow ad = bc$
$\Rightarrow \frac{a}{b} = \frac{c}{d}$

ϕ is onto because any $\frac{f(a)}{f(b)} \in F'$ is the image of $\frac{a}{b} \in F$ under the mapping ϕ

ϕ is homomorphism because

$$\phi\left(\frac{a}{b} + \frac{c}{d}\right) = \phi\left(\frac{ad+bc}{bd}\right) = \frac{f(ad+bc)}{f(bd)}$$

$$= \frac{f(ad) + f(bc)}{\cancel{f(bd)}\, f(b)f(d)}$$

$$= \frac{f(a)f(d) + f(b)f(c)}{f(b)f(d)}$$

$$= \frac{f(a)}{f(b)} + \frac{f(c)}{f(d)}$$

$$= \phi\left(\frac{a}{b}\right) + \phi\left(\frac{c}{d}\right)$$

and $\phi\left(\frac{a}{b}\frac{c}{d}\right) = \phi\left(\frac{ac}{bd}\right) = \frac{f(ac)}{f(bd)}$

$$= \frac{f(a)f(c)}{f(b)f(d)}$$

$$= \frac{f(a)}{f(b)} \frac{f(c)}{f(d)}$$

$$= \phi\left(\frac{a}{b}\right)\phi\left(\frac{c}{d}\right)$$

Therefore ϕ is an isomorphism of F onto F' and hence F is isomorphic to F'.

Thus any two isomorphic integral domains have isomorphic quotient fields.