# Quotient Field

**Def:** If $D$ be an integral domain, then the set $\{\frac{a}{b} : a, b \in D \;\&\; b \neq 0\}$ is called the quotient field of $D$.

**Q.** Prove that any field $K$ containing an integral domain $D$ contains a subfield $K'$ isomorphic to the quotient field $F$ of $D$.

or

Prove that the quotient field $F$ of an integral domain is the smallest field of $D$.

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Let $D$ be an integral domain. Therefore $D$ must be a commutative ring without zero divisors. Let $a \in D$ & $b(\neq 0) \in D$. Since $K$ is a field containing $D$, therefore

$$a \in K,\ b(\neq 0) \in K \Rightarrow ab^{-1} \in K$$

Let $K'$ be the subset of $K$ containing the elements of the form $ab^{-1}$, where $a, b \in D$ with $b \neq 0$.

Thus $K' = \{ab^{-1} \in K ;\ a, b(\neq 0) \in D\}$

We shall show that $K'$ is a subfield of $K$ and $K'$ is isomorphic to quotient field $F$ of $D$.

To show that $K'$ is a subfield of $K$ we have to show that (i) $x, y \in K' \Rightarrow x - y \in K'$

(ii) $x, y \in K'$ & $y \neq 0 \Rightarrow xy^{-1} \in K'$

We have

$x, y \in K' \Rightarrow x = ab^{-1},\ y = cd^{-1}$ for some $a, b, c, d \in D$ such that $b, d \neq 0$.

$$\Rightarrow x - y = ab^{-1} - cd^{-1} = add^{-1}b^{-1} - cbb^{-1}d^{-1}$$
$$= adb^{-1}d^{-1} - cbb^{-1}d^{-1}$$
$$= (ad - cb)\, b^{-1}d^{-1}$$
$$\Rightarrow x - y = (ad - cb)\,(db)^{-1} \in K'$$
$$\Rightarrow x - y \in K'.$$ This proves (i)

Further $x, y \in K'$ and $y \neq 0 \Rightarrow x = ab^{-1},\ y = cd^{-1} \neq 0$ for some $a, b, c, d \in D$ such that $b, d \neq 0$

$$\Rightarrow xy^{-1} = (ab^{-1})(cd^{-1})^{-1} = (ab^{-1})(dc^{-1})$$
$$= ab^{-1}dc^{-1} = ad\,b^{-1}c^{-1}$$
$$\Rightarrow xy^{-1} = (ad)(cb)^{-1} \in K'$$
$$\Rightarrow xy^{-1} \in K'$$

This proves (ii).

We have $F = \{\frac{a}{b} : a \in D, b(\neq 0) \in D\}$, by def$^n$.

Let us now consider a mapping $f : F \to K'$ defined by

$$f\left(\frac{a}{b}\right) = ab^{-1} \quad \forall \frac{a}{b} \in F$$

$f$ is one-one because

$$\begin{aligned} f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right) &\Rightarrow ab^{-1} = cd^{-1} \\ &\Rightarrow ab^{-1}bd = cd^{-1}bd \\ &\Rightarrow ad = cbd^{-1}d \\ &\Rightarrow ad = cb \\ &\Rightarrow \frac{a}{b} = \frac{c}{d} \end{aligned}$$

$f$ is onto because any $ab^{-1} \in K'$ is the image of $\frac{a}{b} \in F$ such that $f\left(\frac{a}{b}\right) = ab^{-1}$

$f$ is homomorphism because

$$\begin{aligned} f\left(\frac{a}{b} + \frac{c}{d}\right) = f\left(\frac{ad+bc}{bd}\right) &= (ad+bc)(bd)^{-1} \\ &= (ad+bc)d^{-1}b^{-1} = add^{-1}b^{-1} + bcd^{-1}b^{-1} \\ &= add^{-1}b^{-1} + cd^{-1}bb^{-1} \\ &= ab^{-1} + cd^{-1} \\ &= f\left(\frac{a}{b}\right) + f\left(\frac{c}{d}\right) \end{aligned}$$

$$\begin{aligned} \& \; f\left(\frac{a}{b} \cdot \frac{c}{d}\right) = f\left(\frac{ac}{bd}\right) &= (ac)(bd)^{-1} \\ &= (ac)(d^{-1}b^{-1}) = (ab^{-1})(cd^{-1}) \\ &= f\left(\frac{a}{b}\right) f\left(\frac{c}{d}\right) \end{aligned}$$

Thus $f : F \to K'$ is a homomorphism & hence

$$F \cong K'$$

If we identify $K'$ with $F$, we see that if $D$ is contained in any field $K$, then $F$ is also contained in $K$. Therefore $F$ is the smallest field containing $D$.