# GROUP THEORY

## QUOTIENT GROUP

Let H be a normal subgroup of G. We denote the set of all right cosets of H in G by $\frac{G}{H}$ i.e.

$$\frac{G}{H} = \{Ha : a \in G\}$$

Let $Ha, Hb \in G/H$. Now we define the operation multiplication on G/H as follows:

$$Ha\, Hb = Hab$$

It can be easily shown that G/H is a group with respect to this operation.

This group G/H is called quotient group.

**Definition**:- If G is a group and H is a normal subgroup of G, then the set G/H of all cosets of H in G is a group with respect to multiplication of cosets, is called quotient group and factor group of G by H.

**Notes**:-

(i) The identity element of the quotient group G/H is H.

**Theorem**:- The set of all cosets of a normal subgroups is a group w.r.t. multiplication of complexes as composition.

**Proof**:- Let H be a normal subgroup of G, so that

$$Hx = xH \; \forall \; x \in G$$

we write $G/H = \{Hx : x \in G\}$

$\Rightarrow$ G/H is a set of cosets of H in G.

Let $e$ be the identity of $G$.
It is obvious that (1)

$$(Ha)(Hb) = Hab \quad \text{——— (1)}$$

Now to show $G/H$ is a group w.r.t. multiplication of complexes as composition.

Let $a, b, c \in G$.

(i) Closure property :- Let $Ha, Hb \in G/H$

$\Rightarrow a, b \in G \Rightarrow a \cdot b \in G$

$\Rightarrow Hab \in G/H \Rightarrow HaHb \in G/H$

(ii) Associativity :- For $(ab)c = a(bc)$, by associativity in $G$.

$\therefore H(ab)c = Ha(bc)$

Using (1) $H(ab) \cdot Hc = Ha \cdot H(bc)$

or, $(Ha\,Hb)Hc = Ha \cdot (Hb \cdot Hc)$.

(iii) Existence of identity :- For $ae = ea = a$ when

$Hae = Hea = Ha$

By (1) $Ha \cdot He = He \cdot Ha = Ha$

$\Rightarrow He = H \in G/H$ is the identity element

(iv) Existence of inverse :-

$a \in G \Rightarrow a^{-1} \in G \Rightarrow Ha^{-1} \in G/H$

and $aa^{-1} = a^{-1}a = e$

$\therefore H(aa^{-1}) = H(a^{-1}a) = He = H$

using (1) $Ha \cdot Ha^{-1} = Ha^{-1} \cdot Ha = H$

$\Rightarrow (Ha)^{-1} = Ha^{-1} \in G/H$.

Hence $G/H$ is a group relative to the operation of multiplication of complexes.

(i) A subgroup H of a group G is normal iff the set G/H of all its left cosets is closed under multiplication.

(ii) Every quotient group of cyclic group is cyclic. but converse is not true.

(iii) Every quotient group of an abilion group G is abelian but converse is not true.

(iv) If H be a normal subgroup of a finite group G. Then $o(G/H) = \frac{o(G)}{o(H)}$

i.e. $o(G/H)$ is the index of H in G.

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