Quadratic form

A second degree homogeneous equation is called a quadratic form.

Exp. (i) $q = ax^2 + 2\sqrt{ab}\,xy + by^2 \Rightarrow$ for two variables
or $ax^2 + 2hxy + by^2$ $\quad h = \sqrt{ab}$ $\quad$ 2-Variables

(ii) $ax^2 + by^2 + cz^2 + 2\sqrt{ab}\,xy + 2\sqrt{bc}\,yz + 2\sqrt{ca}\,zx \to$ 3 Var.
or $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$

(iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2myw$
$+ 2lxw + 2nzw$
for 4-variables.

So that every expression can be written as
$$q = X'AX$$

Theorem – Every quadratic form can be written as $q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = X'AX$

So that the matrix A is always symmetric where $A = [a_{ij}]$

$$X' = [x_1, x_2, \ldots, x_n]$$
$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(I) $ax^2 + by^2 + 2hxy = [x\ y] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(II) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx = X'AX$
$$= [x, y\ z] \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}$$

(III) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx + 2lxw$
$+ 2myw + 2nzw = X'AX \Rightarrow A = \begin{bmatrix} a & h & f & l \\ h & b & g & m \\ f & g & c & n \\ l & m & n & d \end{bmatrix}$

Quadratic form :- Let f be a bilinear form on a vector space V over the field F. then the quadratic form on V associated with the bilinear form f is the function a form V into V defined by $q(u) = f(u,u)$ $\forall$ $u \in V$

$\therefore f(u,v) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i v_j a_{ij}$

An expression of the form

$$q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$ is called a quadratic form in n variables $x_1, x_2, \dots x_n$ where a_{ij}'s are real numbers.

Every quadratic form q can be expressed in matrix form as $q = X^T A X$

where $A = [a_{ij}]_{n\times n}$ is symmetric matrix

X is a column vectors of variables

$X = [x_i]_{n\times 1}$ i.e. $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow X^T = [x_1, x_2 \dots x_n]$

Exp. Given that $q = 2x^2 - y^2 + 2z^2 + 2yz - 4zx + 6xy$

Sol. $q = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{13}xz + 2a_{23}yz + 2a_{12}xy$

$a_{11} = 2$, $a_{22} = -1$, $a_{33} = 2$, $2a_{13} = -4 \Rightarrow a_{13} = -2$

$2a_{23} = 2 \Rightarrow a_{23} = 1$, $2a_{12} = 6 \Rightarrow a_{12} = 3$

where $a_{ij} = a_{ji}$

$$q = [x\ y\ z]\begin{bmatrix} 2 & 3 & -2 \\ 3 & -1 & 1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$a_{13} = a_{31} = -2$

$a_{23} = a_{32} = 1$

$a_{12} = a_{21} = 3$

$q = X^T A X \Rightarrow A = \begin{bmatrix} 2 & 3 & -2 \\ 3 & -1 & 1 \\ -2 & 1 & 2 \end{bmatrix}$

Linear Transformation of a Quadratic form

Let $X'AX$ be quadratic form in n-variables and let $X = PY$, where P is a non-singular matrix

$\Rightarrow X' = (PY)' = Y'P'$

Thus $X'AX = Y'P'AP\cancel{Y}$

$= Y'(P'AP)Y$

$X'AX = Y'BY \Rightarrow X'AX$ is tranformation as $Y'BY$

where $B = P'AP$

Therefore $Y'BY$ is also quadratic form in n-variables

Hence, it is a Linear Transformation of quadratic form $X'AX$ under the L.T. $X = PY$ and $B = P'AP$

Here (i) $B' = (P'AP)' = P'AP = B$

A and B bothe are symmetric matrices

(ii) $\rho(B) = \rho(A) \Rightarrow$ rank of B = rank of A

$\therefore$ A and B are congruent matrices.

Canonical form: If real quadratic form ($X'AX$) be expressed as a sum or difference of the square of new variables by means of any real non-singular linear transformation, then the later quadratic expressed is called a canonical form.

Consider a linear transformation:

If the quadratic form $q = X'AX = \sum_{i=1}^{n} \sum_{j=1} a_{ij} x_i x_j$ can be reduced to the quadratic form $Y'BY = \sum_{i=1}^{n} d_i y_i^2 = [d_1 y_1^2 + d_2 y_2^2 + \dots d_n y_n^2]$ by a non-singular linear transformation $X = PY$, then $Y'BY$ is called the canonical form. $\rightarrow$ canonical form, $\therefore$ λ_i are eigenvalues $\therefore B = P'AP = \text{diag}(d_1 \dots d_n)$

Exp. Reduce the quadratic form :- $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the Canonical form. Also find the matrix of transformation.

Sol. Given quadratic form

$$3x^2+5y^2+3z^2+2zx-2yz-2xy$$
$$a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{31}zx+2a_{23}yz+2a_{12}xy$$

then
$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Its characteristic Equation is

$$|A-\lambda I|=0 \text{ i.e } \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}=0$$

$$|A-\lambda I|=(3-\lambda)\{(5-\lambda)(3-\lambda)-1\}-(-1)\{(-1)(3-\lambda)-1\}+1\{1-(5-\lambda)\}=0$$

or $\lambda^3-S_1\lambda^2+S_2\lambda-|A|=0$

$\lambda^3-11\lambda^2+36\lambda-36=0$

$\lambda=2,3,6$ are eigen value

$|A|=3(15-1)+1(-3+1)+1(1-5)$
$=42-2-4=36$

$S_1=3+5+3=11$

$S_2=\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix}+\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}+\begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$

$S_2=(15-1)+(9-1)+(15-1)$
$14+8+14=36$

Hence the given quadratic form reduces to the Canonical form

$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2$
$=2x^2+3y^2+6z^2$

rank $(r)\rightarrow 3$ (No of non-zero eigen values)

(i) Index $\rightarrow p=3$ (No. of positive terms in canonial form / eigenvalues)

(ii) signature $\rightarrow 3-0=3$ (Diff. b/w (+ve) & (-ve) terms) / eigenvalues

(iii) Nature $\rightarrow$ 3 positive definite (only positive terms (eigenvalues in canonical