(vii) Properties of adjoint and inversion (The most imp. part of matrix)

(a) show that $A(\text{adj} A) = (\text{adj } A)A = |A| I$ for any square matrix A.

Proof :- Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & a_{1n} \\ a_{21} & a_{22} & & a_{2j} & a_{2n} \\ a_{i1} & a_{i2} & & a_{ij} & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nj} & a_{nn} \end{bmatrix}$$

$$\text{Then adj } A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{i1} & A_{n1} \\ A_{12} & A_{22} & & A_{i2} & A_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1j} & A_{2j} & & A_{ij} & A_{nj} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & & A_{in} & A_{nn} \end{bmatrix}$$

Where A_{ij} is the co-factor of a_{ij} in $|A|$.

Since both A and adj A are square matrices.

$\therefore$ The products $A(\text{adj} A)$ and $(\text{adj} A)A$ are both defined.

Now, $(j,i)^{th}$ element in the product $A(\text{adj} A)$.

$$= a_{j1}A_{i1} + a_{j2}A_{i2} + a_{j3}A_{i3} + \cdots + a_{jn}A_{in}$$

$$= \begin{cases} 0 & \text{if } j \neq i \\ |A| & \text{if } j = i \end{cases}$$ (by a well known property of Co-factors)

$$\therefore A\,(\text{adj}\,A) = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= |A|\, I$$

$$(\text{adj}\,A)A = |A|\,I$$

lly we can prove that

Thus $(\text{adj}\,A)A = A(\text{adj}\,A) = |A|\,I$

(b) Prove that the inverse of a square matrix A exists iff A is non-singular [In other words, a necessary and sufficient condition for the existence of the inverse of a square matrix A is that A is non-singular]

Proof :- Necessity

suppose A is a square matrix and its inverse, A^{-1} exists. Let $A^{-1} = B$. Then, by defn. of inverse,
$AB = BA = I$, where I is the unit matrix.

$$\therefore |AB| = |I| = 1$$
$$\text{or, } |A|\,|B| = 1$$
$$\therefore |A| \neq 0, \text{ i.e. } A \text{ is non-singular.}$$

Sufficiency

Let A be non-singular. Then $|A| \neq 0$

Consider the matrix $B = \dfrac{\operatorname{adj} A}{|A|}$

Then $AB = \dfrac{A(\operatorname{adj} A)}{|A|} = \dfrac{|A| I}{|A|} = I$

and $BA = \dfrac{(\operatorname{adj} A)A}{|A|} = \dfrac{|A| I}{|A|} = I$

Thus $AB = BA = I$

$\therefore B$ is the inverse of A

Hence A^{-1} exists and $= \dfrac{\operatorname{adj} A}{|A|}$