Vector Calculus

PRODUCT OF THREE AND FOUR VECTORS

Triple product (i.e. product of three vectors)

If three vectors are combined into a product, the corresponding result is called a triple product. We observe that the following triple products can be formed out of three vectors $\vec{a}, \vec{b}, \vec{c}$

(1) $\vec{a}(\vec{b}\cdot\vec{c})$ (a vector quantity)

Since $\vec{a}$ is a vector and $\vec{b}\cdot\vec{c}$ is a scalar therefore $\vec{a}(\vec{b}\cdot\vec{c})$ is a vector in the direction of $\vec{a}$, whose magnitude is $(\vec{b}\cdot\vec{c})$ times that of $\vec{a}$.

(2) $\vec{a}\cdot(\vec{b}\times\vec{c})$, (a vector quantity)

Since $\vec{a}$ and $\vec{b}\times\vec{c}$ are vectors, therefore scalar product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is a scalar quantity.

(3) $\vec{a}\times(\vec{b}\times\vec{c})$ (a vector quantity)

Since $\vec{a}$ and $\vec{b}\times\vec{c}$ are vectors, therefore their cross product $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector quantity.

(4) $\vec{a} \cdot (\vec{b} \cdot \vec{c})$

(3)

Since $\vec{a}$ is a vector and $\vec{b} \cdot \vec{c}$ is a scalar, therefore their scalar product $\vec{a} \cdot (\vec{b} \cdot \vec{c})$ is absurd, because the scalar product requires both vectors.

(5) $\vec{a} \times (\vec{b} \cdot \vec{c})$

similar argument as in (4)

(6) $\vec{a}\,(\vec{b} \times \vec{c})$ (not defined)

Since $\vec{a}$ and $\vec{b} \times \vec{c}$ are both vectors therefore their operations without dot or cross will be undefined.

Now, $\vec{a} \cdot (\vec{b} \times \vec{c})$ known as scalar triple product and $a \times (b \times c)$ known as the vector triple product.

Scalar triple product

If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ taken in this order is given by $\vec{a} \cdot (\vec{b} \times \vec{c})$.

This involves an ordered triad of vectors and moreover, it is a scalar quantity. we denote it by

$$\boxed{[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})}$$

For example, if $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along three mutually perpendicular axes of Coordinates then

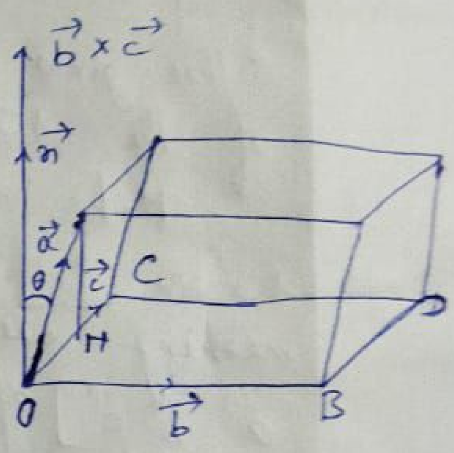
$$[\vec{i}\ \vec{j}\ \vec{k}] = \vec{i}\cdot(\vec{j}\times\vec{k}) = \vec{i}\cdot\vec{i} = 1$$

$$\therefore [\vec{i}\ \vec{j}\ \vec{k}] = 1$$

Geometrical Interpretation of scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$

We show that

$[\vec{a}\ \vec{b}\ \vec{c}]$ = volume of the parallelopiped having concurrent edges $\vec{a}, \vec{b}, \vec{c}$.



Proof: We consider a parallelopiped having concurrent edges OA, OB, OC where O is the origin and take $\vec{OA} = \vec{a}$, $\vec{OB} = b$, $\vec{OC} = \vec{c}$

Now, we have

$$\vec{b}\times\vec{c} = (|\vec{b}\times\vec{c}|)\vec{n} \quad \text{--- (1)}$$

where $|\vec{b}\times\vec{c}|$ = area of the parallelogr OBCD having sides $\vec{b}$ and $\vec{c}$.

Also $\vec{n}$ = unit vector perpendicular to t plane of this parallelogram in the positiv sense of rotation from $\vec{b}$ to $\vec{c}$.

We draw perpendicular to the pla OBDC. If $\vec{a}, \vec{b}, \vec{c}$ form a right handed sy then the angle θ between $\vec{n}$ and $\vec{a}$ is acute i.e $\vec{a}\cdot\vec{n} > 0$.

$\therefore V =$ volume of the parallelopiped
$=$ (height AH) (area of the parallelogram OBCD)

$= (\vec{a}\cdot\vec{n})(|\vec{b}\times\vec{c}|) = \vec{a}\cdot(|\vec{b}\times\vec{c}|\vec{n})$
$= \vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]$ using (1)

Also, when $\vec{a}, \vec{b}, \vec{c}$ form a left handed system, θ is obtuse i.e. $\vec{a}\cdot\vec{n} < 0$. In this case we have $V = -[\vec{a}\ \vec{b}\ \vec{c}]$

Hence we conclude that

$$\boxed{V = \text{Volume of the parallelopiped} = |[\vec{a}\ \vec{b}\ \vec{c}]|}$$

This gives the required geometrical interpretation of the scalar triple product.

Note:- On the basis of definition of the scalar triple product and its geometrical meaning, we see that the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ of three vectors $[\vec{a}, \vec{b}, \vec{c}]$ is a scalar quantity and is numerically equal to the volume V of the parallelopiped of which the three concurrent edges are $\vec{a}, \vec{b}, \vec{c}$ and its sign will be positive or negative according as $\vec{a}, \vec{b}, \vec{c}$ form a right handed or left handed system of vectors.