ATTRACTION AND POTENTIAL

Problems on Equipotential surfaces

Q. 3) show that $(x-c)^2 + y^2 = \lambda[(x+c)^2 + y^2]$ represents a family of equipotential surfaces and that $V = B + A\log\lambda$

we have $\lambda = \dfrac{x^2+y^2+c^2-2cx}{x^2+y^2+c^2+2cx}$

$$\therefore \frac{1+\lambda}{1-\lambda} = \frac{1}{2c}\left(x + \frac{y^2+c^2}{x}\right) \quad \text{——} (1)$$

Differentiating (1) w.r.t. x we get

$$\frac{(1-\lambda)+(1+\lambda)}{(1-\lambda)^2}\frac{\partial\lambda}{\partial x} = \frac{1}{2c}\left\{1 - \frac{y^2+c^2}{x^2}\right\} \text{ or, } \frac{2}{(1-\lambda)^2}\frac{\partial\lambda}{\partial x}$$

$$= \frac{1}{2c}\left\{1 - \frac{y^2+c^2}{x^2}\right\}$$

$$\text{——} (2)$$

$$\left\{\frac{2}{(1-\lambda)^2}\frac{\partial\lambda}{\partial x}\right\}^2 = \frac{1}{4c^2}\left\{\frac{x^2-y^2-c^2}{x^2}\right\}^2$$

$$= \frac{1}{4c^2}\left\{\frac{(x^2+y^2+c^2)^2 - 4x^2(y^2+c^2)}{x^4}\right\}$$

$$= \frac{1}{4c^2}\left\{\frac{4c^2}{x^2}\left(\frac{1+\lambda}{1-\lambda}\right)^2 - \frac{4}{x^2}(y^2+c^2)\right\},$$

by (1)

$$\text{or, } \frac{4}{(1+\lambda)^4}\left(\frac{\partial\lambda}{\partial x}\right)^2 = \frac{1}{c^2x^2}\left\{c^2\left(\frac{1+\lambda}{1-\lambda}\right)^2 - (y^2+c^2)\right\}$$

$$\text{——} (3)$$

Differentiating (2) w.r.t. x we get

$$\frac{4}{(1-\lambda)^3}\left(\frac{\partial\lambda}{\partial x}\right)^2 + \frac{2}{(1-\lambda)^2}\frac{\partial^2\lambda}{\partial x^2} = \frac{1}{2c}\cdot\frac{2}{x^3}(y^2+c^2)$$

or, $$\frac{\partial^2\lambda}{\partial x^2} = \frac{(1-\lambda)^2}{2cx^3}(y^2+c^2) - \frac{2}{(1-\lambda)}\left(\frac{\partial\lambda}{\partial x}\right)^2$$

$$= \frac{1-\lambda}{2cx^3}(y^2+c^2) - \frac{2}{(1-\lambda)}\frac{(1-\lambda)^4}{4c^2x^2}\left\{c^2\left(\frac{1+\lambda}{1-\lambda}\right)^2 - (y^2+c^2)\right\}$$

by (3)

—— (4)

similarly differentiating (1) w.r.t. y we get

$$\frac{2}{(1-\lambda)^2}\frac{\partial\lambda}{\partial y} = \frac{y}{cx} \quad \text{—— (5)}$$

$$\therefore \left(\frac{\partial\lambda}{\partial y}\right)^2 = \frac{(1-\lambda)^4}{y^4}\frac{y^2}{x^2}$$

Differentiating again w.r.t. y

$$\frac{4}{(1-\lambda)^3}\left(\frac{\partial\lambda}{\partial y}\right)^2 + \frac{2}{(1-\lambda)^2}\frac{\partial^2\lambda}{\partial y^2} = \frac{1}{cx}$$

$$\therefore \frac{\partial^2\lambda}{\partial y^2} = \frac{(1-\lambda)^2}{2cx} - \frac{2}{(1-\lambda)}\left(\frac{\partial\lambda}{\partial y}\right)^2$$

$$= (1-\lambda)^2 - \frac{2}{(1-\lambda)}\frac{(1-\lambda)^4}{4c^2}\frac{y^2}{x^2}$$

—— (6)

using 3, 4, 5 and 6 we get

$$\frac{\dfrac{\partial^2\lambda}{\partial x^2}+\dfrac{\partial^2\lambda}{\partial y^2}}{\left(\dfrac{\partial\lambda}{\partial x}\right)^2+\left(\dfrac{\partial\lambda}{\partial y}\right)^2} = \frac{\dfrac{(1-\lambda)^2}{2cx}\left\{1+\dfrac{y^2+c^2}{x^2}\right\}-\dfrac{2}{(1-\lambda)}\dfrac{(1-\lambda)^4}{4c^2x^2}\left\{y^2+c^2\left(\dfrac{1+\lambda}{1-\lambda}\right)^2-y^2-c^2\right\}}{\dfrac{(1-\lambda)^4}{4c^2x^2}\left\{y^2+c^2\left(\dfrac{1+\lambda}{1-\lambda}\right)^2-y^2-c^2\right\}}$$

$$= \frac{2cx}{(1-\lambda)^2}\cdot\frac{x^2+y^2+c^2}{x^2\left\{c^2\left(\dfrac{1+\lambda}{1-\lambda}\right)^2-c^2\right\}}-\frac{2}{1-\lambda}$$

$$= \frac{2cx}{(1-\lambda)^2}\cdot\frac{2cx\left(\dfrac{1+\lambda}{1-\lambda}\right)}{x^2c^2\left\{\left(\dfrac{1+\lambda}{1-\lambda}\right)^2-1\right\}}-\frac{2}{1-\lambda}$$

$$= \frac{4(1+\lambda)}{(1-\lambda)^3}\cdot\frac{(1-\lambda)^2}{(1+\lambda)^2-(1-\lambda)^2}-\frac{2}{1-\lambda}$$

$$= \frac{4(1+\lambda)}{(1-\lambda)\cdot 4\lambda}-\frac{2}{1-\lambda}$$

$$= \frac{1+\lambda-2\lambda}{(1-\lambda)\lambda}$$

$= \frac{1}{\lambda}$ = a function of λ only.

Hence the given family is a possible family of equipotential surfaces.

If V be the potential, we have

$$V = f(\lambda) \text{ where } \frac{f''(\lambda)}{f'(\lambda)} = \frac{-\nabla^2\lambda}{(\nabla\lambda)^2} = -\frac{1}{\lambda}$$

Integrating, $\log f'(\lambda) = \log A - \log\lambda$

or, $f'(\lambda) = \frac{A}{\lambda}$

Integrating again, $f(\lambda) = B + A\log\lambda$

Thus, $V = B + A\log\lambda$