Positive term series

An infinite series in which all the terms are positive is called positive term series.

e.g. $2+7+13+20+\cdots\cdots$

or

An infinite series all of whose terms are positive after some particular term, is positive term series

e.g. $-7-5-2+2+7+13+20+\cdots$

In positive term series as all its terms after the third term (-2) are positive.

⇒ A series of positive terms either converges or diverges to $+\infty$:- For the sum of n terms of positive term series, omitting the negative terms, tends to either a finite limit or $+\infty$.

Consider a positive terms series $\sum a_n$, a_n for all n.

We have $S_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1}$

$S_{n+1} = S_n + a_{n+1}$

$S_{n+1} - S_n = a_{n+1} > 0 \ \forall n$

$\therefore S_{n+1} > S_n \ \forall n$ and so the sequence $\langle S_n \rangle$ of partial sum of $\sum a_n$ is monotonically increasing. Hence $\sum a_n$ is convergent if and only if it is bounded above

(1) A positive term series $\sum a_n$ is convergent if and only if its sequence $\langle S_n \rangle$ of partial sums is bounded above. i.e $\sum a_n$ converges $\Leftrightarrow S_n < k \ \forall n$. k being some positive real numbers.

(2) If $\sum a_n$ is a positive term series such that

$$\lim_{n \to \infty} a_n \neq 0, \text{ then } \sum a_n \text{ is divergent.}$$

Remark → We know that a monotonic (increasing or decreasing) sequence can converge or diverge but cannot oscillate.

Comparison tests :-

If two positive term series $\sum a_n$ and $\sum b_n$ be such that

(i) $\sum b_n$ converges

(ii) $a_n \leq k b_n \quad \forall \ n \geq m$

k is a fixed positive number and m a fixed positive integer.

Then (i) $\sum b_n$ converges $\Rightarrow \sum a_n$ converges

(ii) $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges

Proof- (i) Since $\sum b_n$ is convergent and $a_n \leq b_n \ \forall n$

$\therefore \lim_{n \to \infty} (b_1 + b_2 + b_3 + \cdots + b_n) =$ a finite quantity $= k$

Also since $a_1 \leq b_1$

$a_2 \leq b_2$

$\cdots$

$\cdots$

$a_n \leq b_n$

Adding

$a_1 + a_2 + a_3 + \cdots a_n \leq b_1 + b_2 + b_3 + \cdots + b_n$

$\lim_{n \to \infty} (a_1 + a_2 + a_3 + \cdots a_n) \leq \lim_{n \to \infty} (b_1 + b_2 + b_3 + \cdots + b_n) = k.$

Hence the series $\sum a_n$ also converges.

(ii) $\sum b_n$ diverges and $a_n \geq b_n \ \forall n.$

then $\sum a_n$ also diverges.

Suppose $\sum b_n$ diverges

then $\lim_{n \to \infty} T_n = +\infty$

The sequence $\langle T_n \rangle$ of partial sums of the series $\sum b_n$ diverges and so $\sum a_n$ diverges.

Exp. Test for convergence the series

$$1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots\cdots$$

Solution. Given series

$$1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots\cdots$$

We notice that $n! \geq 2^{n-1} \quad \forall\ n\geq 2$

$$\therefore \frac{1}{n!} \leq \frac{1}{2^{n-1}} \quad \forall\ n\geq 2$$

Now $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots$ being geometric series with common rati $\frac{1}{2}<1$ is convergent.

Hence, by first comparison test, $\sum\frac{1}{n!}$ convergent.

Exp. Test for convergence the series

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots\cdots\frac{1}{n^n}+\cdots$$

Sol. Clearly, $n^n > 2^n$, for $n>2$

$$\frac{1}{n^n} < \frac{1}{2^n} \quad \text{for } n>2$$

Since $\sum \frac{1}{2^n} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$ is a geometric series with common ratio $\frac{1}{2}<1$, so $\sum \frac{1}{2^n}$ is convergent. Hence by comparision test, $\sum \frac{1}{n^n}$ is convergent.